Selecting with Repetition but without Order
Balls in Bins

We know that there are $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ ways to select *k* objects from *n* with order ignored but allowing repetition. Here is a way to see it using balls and bins.

Think of *n* bins labeled $a_1, a_2, ..., a_n$. I want to throw *k* balls into the bins. In the end I care about how many balls are in each bin but I do not care which ball went into which bin (i.e. if the first bin ends up with two balls, I do not care whether they were balls 3 and 5 or 2 and 8 or whatever. All I care about is that there are two).

Think of the number of balls in bin a_1 as the number of times a_1 was selected, then notice that this is selecting allowing repetition (more than one ball may be in the a_1 bin) but without order (since I don't care which balls are in the a_1 bin).

Consider a little diagram looking like this

$$\circ\circ|\circ\ |||\circ\circ\circ$$

The strokes | suggest the divisions between the bins. Thus this diagram represents two balls in bin a_1, one in bin a_2, none in a_3 and a_4, and three in a_5.

So how many such diagrams could I construct with six balls (i.e. the little $\circ$'s) and four strokes (one less than the number of elements $a_1, a_2, ..., a_5$)? Alternatively said: How many ways of putting six balls in five bins? Alternatively said: How many ways of selecting six items from five allowing repetition and ignoring order?

With six balls and four strokes, that's a total of ten objects. If you mark which ones of the ten slots must contain $\circ$'s, the rest is fixed (since I just fill strokes into the empty slots). So there are $\binom{10}{6}$ ways of putting the six balls into the five bins.

In general, you have slots for the *k* $\circ$s and slots for $n-1$ |s for a total of $n+k-1$. Of these you select the *k* to contain the $\circ$s. (That's $\binom{n+k-1}{k}$ ways.) Alternatively, you select the $n-1$ to contain the |s for $\binom{n+k-1}{n-1}$ ways. The selection is fixed after the $\circ$s (or the |s) have been positioned. You read off exactly how many balls in each bin.

Example: There are $\binom{6+10-1}{10}$ ways of rolling ten six-sided dice since this is equivalent to throwing ten balls into six bins.