1. Present a combinatorial argument that for all positive integers *x* and *y*

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n .$$

(Hint: Consider sequences drawn from the union of distinct sets A and B of cardinalities *x* and *y*, respectively.)

2. Present a combinatorial argument that for all positive integers $1 \le k \le m \le r$:

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

3. a. Present a combinatorial argument that for all positive integers *n*, *a*, and *b(>a)*:

$$\sum_{k=0}^{n} \binom{n}{k} a^k (b-a)^{n-k} = b^n .$$

b. Present a combinatorial argument that for all positive integers *n:*

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

4. Using a combinatorial argument, prove that for $n \ge 1$:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

5. a. Present a combinatorial argument that for all positive values of *n*:

$$3^n = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i \quad j}$$

b. Present a combinatorial argument that for all *m* and *n* satisfying $2 \le m$, $2 \le n$, and $m \le n+1$:

$$\binom{n+2}{m} = \binom{n+1}{m} + \binom{n}{m-1} + \binom{n}{m-2}$$

(Hint: Consider $A = B \cup \{c\} \cup \{d\}$, where $c \ne d$, $c \notin B, d \notin B$, and $\# B = n$.)

6. a. Present a combinatorial argument that for all *n* and *k* satisfying $1 \le n$ and $k \le n$:

$$n! = \binom{n}{k} \cdot k! \cdot (n-k)!$$

b. Present a combinatorial argument that for all positive values of *n*:

$$2^n = 1 + \sum_{k=0}^{n-1} 2^k$$

(Hint: Consider Let *k* be the position of the first 1 in a bit string.)

7. a. Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

b. Present a combinatorial argument that for all nonegative integers *p*, *s*, and *n* satisfying $p + s \leq n$

$$\binom{n}{p}\binom{n-p}{s} = \binom{n}{p+s}\binom{p+s}{p}$$

(Hint: Consider choosing two subsets.)

8. a. Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

(**Note**: The summation begins with $k = 1$.)

b. Present a combinatorial argument that for all integers *k* and *n* satisfying $3 \leq k \leq n$

$$\binom{n}{k} = \binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$$

(**Hint:** Consider three special elements.)

9. Present a combinatorial argument that for all positive integers $m, n,$ and r, satisfying $r \leq \min\{m, n\}$:

$$\binom{m+n}{r} = \sum_{k=0}^{n} \binom{m}{k}\binom{n}{r-k}.$$

(**Hint**: Consider selecting from two sets.)

b. Present a combinatorial argument that for all positive integers n:

$$3^n = \sum_{i=0}^{n} \left(\sum_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} \right)$$

(**Note:** Be very specific about the roles of i and j.)