4. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Let A and B be disjoint sets of cardinality n each and $C = A \cup B$. How many subsets of C are there of cardinality n. We are selecting elements for such a subset without repletion not with concern for order so there are $\binom{2n}{n}$ such subsets. Alternatively, let k represent the number of elements in such a subset that were selected from A. The value of k may vary from 0 to n. There are $\binom{n}{k}$ such selections of the k elements from A. Now select which k elements from B will **not** be in the subset (the k that remain will thus be **in** the subset). There are $\binom{n}{k}$ of selecting these so $\binom{n}{k}^2$ ways of selecting the subset and $\sum_{k=0}^{n} \binom{n}{k}^2$ ways overall. This must equal $\binom{2n}{n}$.

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

Let $A = \{a, b, c\}$ and consider all strings of length n using elements of A. Since there are three options for each component of the string, there are 3^n such strings. Alternatively, consider first consider the positions of any *c*'s in the string. Let *k* represent the number of non-*c*'s (i.e., *a*'s and *b*'s) in the string. Clearly *k* could range from 0 through *n*. For a fixed value of *k*, there are $\binom{n}{k}$ ways to choose the positions for the non-*c*'s . Then for each of the *k* positions, there are two options (i.e., *a* or *b*) for the character in the position. The remaining *n-k* positions must be occupied by *c*'s. Thus there are $\binom{n}{k} 2^k$ ways to assign elements to the positions with *k* non-*c*'s. The total is $\sum_{k=0}^{n} \binom{n}{k} 2^k$ and this must equal 3^n

b. [10] Present a combinatorial argument that for all nonnegative integers *p*, *s*, and *n* satisfying $p + s \leq n$

$$\binom{n}{p}\binom{n-p}{s} = \binom{n}{p+s}\binom{p+s}{p}$$

(Hint: Consider choosing two subsets.)

Let a set *A* have *n* elements and consider how many ways there are to select disjoint subsets *B* and *C* of *A* so that *B* has p elements and *C* has *s* elements. First we could select the *p* elements for *B* in $\binom{n}{p}$ ways and then select the *s* elements for *C* from the remaining *n-p* elements of $A \sim B$ in $\binom{n-p}{s}$ ways. Together this yields $\binom{n}{p}\binom{n-p}{s}$ such selections. Alternatively, we could first select the *p*+*s* elements for $B \cup C$ in $\binom{n}{p+s}$ ways and then select the *p* elements for *B* from $B \cup C$ in $\binom{p+s}{p}$ ways. There are thus $\binom{n}{p+s}\binom{p+s}{p}$ such selections and this must equal $\binom{n}{p}\binom{n-p}{s}$

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$\sum_{k=1}^{n}\binom{n}{k} = 2^n - 1$$

(**Note**: The summation begins with $k = 1$.)

Consider the cardinality of the set of non-empty subsets of a set A of *n* elements. For each element of A, there are two options: either be present in a subset or not. Thus there are 2^n total subsets but one of these is empty so there are $2^n - 1$ non-empty subsets of A. Alternatively, let *k* indicate the cardinality of the subset. Since we are counting non-empty subsets, *k* ranges from 1 to *n*. For a fixed value of *k*, there are $\binom{n}{k}$ ways of selecting the *k* subset elements from the *n* total elements of A. Adding this to include all possible cases of k, we obtain $\sum_{k=1}^{n}\binom{n}{k}$ and this must equal $2^n - 1$.

b. [10] Present a combinatorial argument that for all integers *k* and *n* satisfying $3 \le k \le n$

$$\binom{n}{k} = \binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$$

(**Hint:** Consider three special elements.)

Consider the number of subsets of size *k* of a set B of cardinality *n*. Since $n \ge 3$, we may select three elements b_1, b_2, b_3 of B and let C = B ~ $\{b_1, b_2, b_3\}$. Thus C has cardinality n-3 and B = C $\cup \{b_1, b_2, b_3\}$. We know there are $\binom{n}{k}$ such subsets. Alternatively, to select *k* elements of B for a subset there are four options: all k come from C, *k*-1 come from C and the *k*th is either $b_1, b_2,$ or b_3, *k*-2come from C and the k-1st and *k*th are exactly two of $b_1, b_2,$ or b_3, or *k*-3 come from C and all of $b_1, b_2,$ and b_3 are present. For the first option, there are $\binom{n-3}{k}$ possibilities since all *k* come from C. For the second option, there are $3\binom{n-3}{k-1}$ possibilities, since *k*-1 elements are selected from C and one from the three of $b_1, b_2,$ or b_3. For the third option, there are $3\binom{n-3}{k-2}$ possibilities, since *k*-2 elements are selected from C and one from the three of $b_1, b_2,$ or b_3 is **not** selected. Lastly, if *k*-3 come from C and all of $b_1, b_2,$ and b_3 are present, then there are $\binom{n-3}{k-3}$ options. The total is $\binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$ and this must equal $\binom{n}{k}$

3. [10] Present a combinatorial argument that for all positive integers $m, n,$ and r, satisfying $r \le \min\{m, n\}$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

(**Hint**: Consider selecting from two sets.)

Let A and B be disjoint sets of cardinalities m and n, respectively. Let $C = A \cup B$ and consider the number of subsets of C of cardinality r. Since $|C| = |A| + |B) = m + n$, there are $\binom{m+n}{r}$ such subsets. Alternatively let k be the number of elements in a subset that came from A. The value of k can range from 0 to r. For a fixed value of k, there are $\binom{m}{k}$ ways to select the k elements from

A and $\binom{n}{r-k}$ ways to select the remaining $r-k$ elements from B, thus $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ total ways. This must equal $\binom{m+n}{r}$.

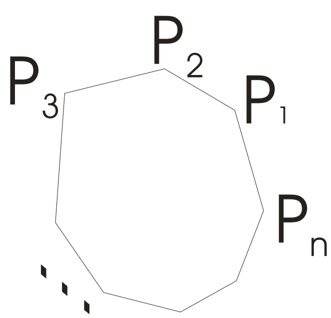
3. [10] Present a combinatorial argument that for all positive integers n :

$$3^n = \sum_{k=0}^{n}\binom{n}{k}2^k .$$

Consider as a model strings of length n using the characters from the set $\{a,b,c\}$. For each n positions there are 3 options so there are 3^n such strings. Alternatively, let k represent the number of positions in the string not occupied by a (i.e., thus, occupied by either b or c). The value of k can vary between 0 and n. For a fixed number k of b s and c s, there are $\binom{n}{k}$ ways to determine the positions to be occupied by the b s and c s and then 2 choices (either b or c) for each of these k positions, for a total of $\binom{n}{k}2^k$ possibilities. The remaining $n-k$ positions must be occupied by a s. Summing over all possible values of k. We have $\sum_{k=0}^{n}\binom{n}{k}2^k$ such strings and this must equal 3^n.

Examination 1 Solutions
CS 336

1. [5] For $n \geq 3$, how many diagonals does a convex polygon with *n* extreme points have? (Consider a convex polygon given by extreme points $< P_1, P_2, \ldots, P_n >$ in counterclockwise order A "diagonal" is a line segment connecting two **non-adjacent** extreme points.)



For each of the *n* extreme points there are *n-3* distinct extreme points that non-adjacent. This would yield $n(n-3)$ endpoints of the diagonals. Since each diagonal has two endpoints, there are $\frac{n(n-3)}{2}$ diagonals of a convex polygon with *n* extreme points.

2. a. [10] Present a combinatorial argument that for all $n \geq 1$:

$$(2n-1)\cdot(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{n!2^n}$$

Consider the set of all partitions of a set of cardinality $2n$ into *n* pairs. For the left side, begin with any permutation of the $2n$ elements. The first element on the permutation is in some pair and there are $2n-1$ choices for its pair-mate. Removing these two from the permutation, the next element permutation is also in some pair and there are $2n-3$ choices for its pair-mate. The process continues until there are just two elements left in the permutation, and they form the last pair. This yields $(2n-1)\cdot(2n-3)\cdots 3\cdot 1$ different such partitions. Now consider the right hand side. There are $(2n)!$ different permutations of the of the $2n$ elements. Pair the first element with the second, the third with the fourth, etc. This yields a partition into *n* pairs. However, the order among the *n* pairs is irrelevant to the partition and thus for every array of pairs there are 2^n different permutations. Lastly, the order among the pairs, is also irrelevant, so a set of pairs could be arranged in $n!$ different orders. Thus the number of partitions into pairs that ignores order within and among pairs is $\frac{(2n)!}{n!2^n}$ and this must equal $(2n-1)\cdot(2n-3)\cdots 3\cdot 1$.

b. [10] Present a combinatorial argument that for all nonegative integers *k* and *n* satisfying $k \le n-2$

$$\binom{n+2}{k} = \binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$$

Let set A have cardinality n and b and c be distinct elements not contained in A. Consider the subsets of $A \cup \{b\} \cup \{c\}$ of cardinality k. For the left hand side, we recognize that $A \cup \{b\} \cup \{c\}$ has cardinality $n+2$, so there are $\binom{n+2}{k}$ such subsets. Alternatively, consider that a subset wither has all k elements coming from A, exactly $k-1$ elements coming from A, or A, exactly $k-2$ elements coming from A. If all k elements come from A, there are $\binom{n}{k}$. If exactly $k-1$ elements come from A, there are $\binom{n}{k-1}$ ways to select those elements and then two choices, b or , to complete the subset. If exactly $k-2$ elements come from A, there are $\binom{n}{k-2}$ ways to select those elements and then both b and must be selected to complete the subset. The total is $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2}$ and this must equal $\binom{n+2}{k}$.

3. Present a combinatorial argument that for all positive values of *m*, *n*, and *r*:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Consider distinct sets A and B of cardinalities *m* and *n*, respectively. There are $\binom{m+n}{r}$ subset of $A \cup B$ of size *r*. Alternatively, for any such subset, there must be some *r-k* elements of A and *k* elements of B for a value of *k* between 0 and *r*. For a fixed *k* there are $\binom{m}{r-k}\binom{n}{k}$ such subsets and thus $\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$ overall.

3. a. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} \binom{n}{n-k} = n \binom{2n-1}{n-1}$$

(Hint: Let A and B be disjoint sets of cardinality *n*. Consider pairs <*C, a*> where $C \subseteq A \cup B$, C has cardinality *n*, and $a \in C \cap A$.)

Using the notation of the hint, first choose a and then choose $C \sim \{a\}$. There are *n* choices for a (since $\# A = n$) and there remain *2n-1* elements in $A \cup B \sim \{a\}$. Thus, there are $n\binom{2n-1}{n-1}$ total choices. Alternatively, let $k = \#(A \cap C)$. The value of *k* can range from *1* (since $a \in A \cap C$) to *n*. For a fixed k, there are $\binom{n}{k}$ choices for $A \cap C$, *k* choices from that for a, and $\binom{n}{n-k}$ choices for $C \cap B$. The total is $\sum_{k=1}^{n} k \binom{n}{k}\binom{n}{n-k}$

b. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Using the same notation as above, consider choosing just a set $C \subseteq A \cup B$ of cardinality *n*, There are $\binom{2n}{n}$ such choices. Alternatively, let *k* be the number of elements in $A \cap C$: k can range from *0* to *n*. For a fixed k, there are $\binom{n}{k}$ ways of choosing $A \cap C$, and since there are *n-k* elements in $B \cap C$ there are *k* elements in $B \sim C$, and hence $\binom{n}{k}$ ways of choosing them. Choosing $B \sim C$ however is equivalent to choosing

$B \cap C$ and thus there are $\binom{n}{k}$ ways to choose $B \cap C$. The total is $\sum_{k=0}^{n} \binom{n}{k}^2$ and this must equal $\binom{2n}{n}$.

3. a. [10] Using a combinatorial argument, prove that for $n \geq 2$ and $m \geq 2$:

$$\binom{n+m}{2} = n \cdot m + \binom{n}{2} + \binom{m}{2}$$

Let A and B be disjoint sets of cardinalities n and m, respectively. We seek to determine how many subsets of two elements there are in $A \cup B$. Since the cardinality of $A \cup B$ is $n+m$, there are $\binom{n+m}{2}$ such subsets. Alternatively, we could obtain such a subset by selecting one element from each of A and B, by selecting both elements from A, or by selecting both elements from B. There are $nm + \binom{n}{2} + \binom{m}{2}$ ways of doing this and, therefore $\binom{n+m}{2} = nm + \binom{n}{2} + \binom{m}{2}$.

b. [10] Using a combinatorial argument, prove that for integers $m, n, p \geq 1$:

$$(n+m)^p = \sum_{k=0}^{p} \binom{p}{k} n^k \, m^{p-k}$$

Let A and B be disjoint sets of cardinalities n and m, respectively. We seek to determine how many strings of length p there are consisting of elements of $A \cup B$. Since the cardinality of $A \cup B$ is $n+m$, there are $n+m$ options for each of p positions in the sequence, so there are $(n+m)^p$ such sequences. Alternatively, let k denote the number of positions in the sequence occupied by elements of A. The value of k varies from 0 to p. For a fixed value of k, there are $\binom{p}{k}$ ways to select these positions and then n options for each of the k positions. For each of the $p-k$ positions occupied by elements of B, there are m options, thus $\binom{p}{k} n^k \, m^{p-k}$ for the fixed value of k and $\sum_{k=0}^{p} \binom{p}{k} n^k \, m^{p-k}$ overall. This must equal $(n+m)^p$.

3. a. [10] Using a combinatorial argument, prove that for $n \geq 2$ and $m \geq 2$:

$$\binom{n+m}{2} = n \cdot m + \binom{n}{2} + \binom{m}{2}$$

Consider subsets of two elements from the union of disjoint subsets A and B with cardinalities n and m, respectively. Since $\#(A \cup B) = n + m$, there are $\binom{n+m}{2}$ subsets of size two. Alternatively, consider that either one element comes from each of A and B, both from A, or both from B. These can be done in $n \cdot m, \binom{n}{2}$, and $\binom{m}{2}$ ways, respectively, and the total is $n \cdot m + \binom{n}{2} + \binom{m}{2}$. We conclude that $\binom{n+m}{2} = n \cdot m + \binom{n}{2} + \binom{m}{2}$

b. [10] Using a combinatorial argument, prove that for $n \geq 1$:

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

(Hint: Let *A* be a set of cardinality *n*. Consider pairs <*B, a*> where $B \subseteq A$ and $a \in A \sim B$.)

Employing the notation from the hint, and considering the left side of the equation first, there are n choices for a and then 2^{n-1} subsets from the remaining $n-1$ elements. Alternatively, let k be the number of elements in $\{a\} \cup B$. The value of k could range from 1 through n. For a fixed value of k, there are $\binom{n}{k}$ ways to choose $\{a\} \cup B$, and then k choices from this for a (with the remaining chosen elements forming B). There are $\sum_{k=1}^{n} k \binom{n}{k}$ total ways of doing this and this must equal $n2^{n-1}$.

3. **a. [10]** Using a combinatorial argument, prove that for $n \geq 1$ and $m \geq 2$:

$$\sum_{k=0}^{n} \binom{n}{k} (m-1)^{n-k} = m^n$$

Consider strings of length n selected from the integers $\{1, 2, ..., m\}$ with repetition allowed. For each of n positions there are m choices, so there are m^n such strings. Alternatively, let k indicate the number of copies of m in the string. The value of k varies from 0 to n. For a fixed value of k there are $\binom{n}{k}$ selections for the placement of the m s and then $(m-1)$ choices for the integers $\{1, 2, ..., m-1\}$ in each of the $n-k$ remaining positions. Thus there are $\binom{n}{k}(m-1)^{n-k}$ such strings with k copies of m, and $\sum_{k=0}^{n}\binom{n}{k}(m-1)^{n-k}$ overall. This must equal m^n.

b. [10] Using a combinatorial argument, prove that for $n \geq k \geq 0$:

$$\binom{n}{k}k!(n-k)! = n!$$

Consider permutations of length n selected from the integers $\{1,2,...,n\}$. There are $n!$ such permutations. Alternatively, let k satisfy $n \geq k \geq 0$ and for any permutation first select the positions to be occupied by $\{1,2,...,k\}$. There are $\binom{n}{k}$ such selections. Now permute the values $\{1,2,...,k\}$ - there are $k!$ such permutations. Finally, permute the $n-k$ values $\{k+1,k+2,...,n\}$, which can be done in $(n-k)!$ ways, and place them into the positions of the permutation notoccupied by the values from $\{1,2,...,k\}$. Thus, there are $\binom{n}{k}k!(n-k)!$ such permutations and this must equal $n!$.