Is the converse of the theorem:
If there exists a function $f:[0,1]\xrightarrow{1-1}A$, then A is uncountably infinite,
true?

That is, if A is uncountably infinite must there exist a function $f:[0,1]\xrightarrow{1-1}A$?

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The answer is "it depends". That's a rare answer in logic or mathematics, but it does apply here. If your appetite is now whetted, please hold on while I put the problem into some context. Then I will explain "it depends".

A finite set of k objects can always be put into one-to-one correspondence with $\{1,,2,\ldots,k\}$. That's even the definition.

A countably infinite set can always be put into one-to-one correspondence with the set of natural numbers. That's the definition, too, but there are some countably infinite sets that ***seem*** much larger (in a cardinality sense) than the natural numbers. Consider first Q the set of all rational numbers (i.e. ratios of integers to non-zero integers). Already it seems that natural numbers are cardinally smaller - but they aren't. Now consider for a fixed k the set Q^k, the set of all k-tuples of rational numbers. That seems still larger – but it's not. Lastly, consider $\bigcup_{k=1}^{\infty} Q^k$, the set of ***all*** k-tuples of rationals ***for all k***. That seems really huge. Yet it is countably infinite and, from a cardinality point of view, is the same as just the set of natural numbers. Put in other words, I can devise a counting that eventually hits any given k-tuple of rationals.

One might wonder "Is there such a canonical set for uncountably infinite sets?". Is there some magic set C such that all uncountably infinite sets can be put into one-to-one correspondence with C? We already know the answer is no, since we know C and $P(C)$, the power set of C, cannot be put into one-to-one correspondence with one another.

Well, then let's relax the rule a bit: we know that we cannot guarantee to find a function f so that $f:C\xrightarrow[onto]{1-1}A$ but can we find a function f so that $f:C\xrightarrow{1-1}A$? By dropping the onto requirement we are asking (loosely) "Is there some cardinally minimum uncountably infinite set C"? The question above is more specific "Is $[0,1]$ such a set", that is:

If A is uncountably infinite must there exist a function $f:[0,1]\xrightarrow{1-1}A$?

and you see we are back to the original question from above.

So now recall my equivocal answer was "it depends". Here (finally) is what I meant. Assuming that this is true or that it is not true - either way - does not violate the axioms of our set theory. ***You can have it either way.*** (Suppose you have only one axiom in life other than those of first-order logic. That one axiom is "NCIS is the worst show on television.". Could you add the axiom that "All Martians have three ears." ? Could you add the axiom that "Some Martians do not have three ears."

? The answer is either will work. You do not violate “NCIS is the worst show on television.” by assuming “All Martians have three ears.” or by assuming “Some Martians do not have three ears.”.)

So, is that it? Is there nothing more to say than “You can have it either way”? There is more. Most folks who do cardinality theory add what is called the Continuum Hypothesis to set theory. That says that there is no set with a cardinality between that of the natural numbers and that of the interval $[0,1]$. So, if we accept the Continuum Hypothesis, the interval $[0,1]$ actually ***is*** a minimal cardinality uncountably infinite set and thus we have the original theorem in both directions:

There exists a function $f:[0,1]\xrightarrow{1\text{-}1}A$ if and only if A is uncountably infinite.

For more on the Continuum Hypothesis, its history, its generalization, and why it’s called that you might read

http://en.wikipedia.org/wiki/Continuum_hypothesis