**Theorem:** For $p > 0$ and $b > 1$, $n^p = o(b^n)$.

**Proof:**

(This proof assumes that $b > 2$. It is left as an exercise how to alter it for $b > 1$.) Let $\bar{b} = b / 2$. We know that since $\bar{b} > 1$, $\log_{\bar{b}} n = o(n)$. That says that given $\varepsilon > 0$, there exists an $\bar{N}$ so that for $n \geq \bar{N}$, $|\log_{\bar{b}} n| \leq \varepsilon |n|$. In particular, let $\varepsilon = \frac{1}{p}$ and $\bar{N}$ be such that for $n \geq \bar{N}$, $|\log_{\bar{b}} n| \leq \frac{1}{p} |n|$.

Notice then that $n \leq \bar{b}^{\varepsilon n}$ and $|n^p| = n^p \leq \bar{b}^{p \varepsilon n} = \bar{b}^n = \left( \frac{b}{2} \right)^n = \frac{b^n}{2^n} = \frac{1}{2^n} |b^n|$. Given $\varepsilon > 0$, select $N = \max\{ \bar{N}, -\log_2 \varepsilon \}$. For $n \geq N$, we have both $n \geq \bar{N}$ and $n \geq -\log_2 \varepsilon$, so, in particular, $\frac{1}{2^n} \leq \varepsilon$.

Combining this with above we have, for all $n \geq N$, $|n^p| \leq \varepsilon |b^n|$.

**Discovery:**

What is above is the proof and no more need be said unless one wonders how the proof was created. To see that imagine that I began with using $\log_{\bar{b}} n = o(n)$, thus knowing that for some $\bar{N} n \geq \bar{N}$ guarantees that $|\log_{\bar{b}} n| \leq \varepsilon |n|$. At that point I did not know how to relate $\bar{N}, \varepsilon$, and $\bar{b}$ to the parameters of the original item to prove. That is, how are $\bar{N}, \varepsilon$, and $\bar{b}$ related to $N, \varepsilon, b$, and $p$ when I attempt to conclude that for any $\varepsilon > 0$, there is an $N$ such that for all $n \geq N$, $|n^p| \leq \varepsilon |b^n|$? I had to try to make one look like the other. This is scratch paper time. Let’s dispense with absolute values for the moment – we can confirm later if that’s ok but they just clutter things on scratch paper for now.

We have $\log_{\bar{b}} n \leq \varepsilon n$ so that can be transformed to $n \leq \bar{b}^{\varepsilon n}$ and to $n^p \leq (\bar{b}^{\varepsilon n})^p = \bar{b}^{p \varepsilon n}$, which is beginning to look like what I need. What I’d like is to relate $\varepsilon$ and $\bar{b}$ to $\varepsilon, b$, and $p$ in such a way that $n^p \leq \bar{b}^{p \varepsilon n}$ looks like $n^p \leq \varepsilon b^n$. First I set $\varepsilon = \frac{1}{p}$. That gives me $n^p \leq \bar{b}^n$. Now I have to construct a relation between $\bar{b}, b$, and $\varepsilon$ so that $\bar{b}^n \leq \varepsilon b^n$. The chain of inequalities would then give me $n^p \leq \bar{b}^n \leq \varepsilon b^n$ which is what I need. But that just says I need $\frac{\bar{b}^n}{b^n} \leq \varepsilon$ which is easy for sufficiently large $n$ if $\frac{\bar{b}}{b} = \frac{1}{2}$. That’s equivalent to having $\bar{b} = b / 2$ and the “sufficiently large $n$” is satisfied if $\frac{1}{2^n} \leq \varepsilon$. That means $n \geq -\log_2 \varepsilon$. The argument reverses and results in the proof above.
(Hint for constructing the proof for $b > 1$: Can you relate $\bar{b}, b,$ and $\varepsilon$ in such a way that you can still make $\frac{\bar{b}^n}{b^n} \leq \varepsilon$ for sufficiently large values of $n$?)