Theorem: For $p > 0$ and $b > 1$, $n^p = o(b^n)$.

Proof:

(This proof assumes that $b > 2$. It is left as an exercise how to alter it for $b > 1$.) Let $\bar{b} = b/2$. We know that since $\bar{b} > 1$, $\log_{\bar{b}} n = o(n)$. That says that given $\bar{\varepsilon} > 0$, there exists an $\bar{N}$ so that for $n \geq \bar{N}$, $|\log_{\bar{b}} n| \leq \bar{\varepsilon}\,|n|$. In particular, let $\bar{\varepsilon} = \frac{1}{p}$ and $\bar{N}$ be such that for $n \geq \bar{N}$, $|\log_{\bar{b}} n| \leq \frac{1}{p}|n|$. Notice then that $n \leq \bar{b}^{\bar{\varepsilon} n}$ and $|n^p| = n^p \leq \bar{b}^{\bar{\varepsilon} pn} = \bar{b}^n = \left(\frac{b}{2}\right)^n = \frac{b^n}{2^n} = \frac{1}{2^n}|b^n|$. Given $\varepsilon > 0$, select $N = \max\{\bar{N}, -\log_2 \varepsilon\}$. For $n \geq N$, we have both $n \geq \bar{N}$ and $n \geq -\log_2 \varepsilon$, so, in particular, $\frac{1}{2^n} \leq \varepsilon$. Combining this with above we have, for all $n \geq N$, $|n^p| \leq \varepsilon\,|b^n|$.

Discovery:

What is above is the proof and no more need be said unless one wonders how the proof was created. To see that imagine that I began with using $\log_{\bar{b}} n = o(n)$, thus knowing that for some $\bar{N}$ $n \geq \bar{N}$ guarantees that $|\log_{\bar{b}} n| \leq \bar{\varepsilon}\,|n|$. At that point I did not know how to relate $\bar{N}, \bar{\varepsilon}$, and $\bar{b}$ to the parameters of the original item to prove. That is, how are $\bar{N}, \bar{\varepsilon}$, and $\bar{b}$ related to N, ε, b, and p when I attempt to conclude that for any $\varepsilon > 0$, there is an N such that for all $n \geq N$, $|n^p| \leq \varepsilon\,|b^n|$? I had to try to make one look like the other. This is scratch paper time. Let's dispense with absolute values for the moment – we can confirm later if that's ok but they just clutter things on scratch paper for now.

We have $\log_{\bar{b}} n \leq \bar{\varepsilon} n$ so that can be transformed to $n \leq \bar{b}^{\bar{\varepsilon} n}$ and to $n^p \leq \left(\bar{b}^{\bar{\varepsilon} n}\right)^p = \bar{b}^{p\bar{\varepsilon} n}$, which is beginning to look like what I need. What I'd like is to relate $\bar{\varepsilon}$ and $\bar{b}$ to ε, b, and p in such a way that $n^p \leq \bar{b}^{p\bar{\varepsilon} n}$ looks like $n^p \leq \varepsilon b^n$. First I set $\bar{\varepsilon} = \frac{1}{p}$. That gives me $n^p \leq \bar{b}^n$. Now I have to construct a relation between $\bar{b}, b$, and ε so that $\bar{b}^n \leq \varepsilon b^n$. The chain of inequalities would then give me $n^p \leq \bar{b}^n \leq \varepsilon b^n$ which is what I need. But that just says I need $\frac{\bar{b}^n}{b^n} \leq \varepsilon$ which is easy for sufficiently large n if $\frac{\bar{b}}{b} = \frac{1}{2}$. That's equivalent to having $\bar{b} = b/2$ and the "sufficiently large n" is satisfied if $\frac{1}{2^n} \leq \varepsilon$. That means $n \geq -\log_2 \varepsilon$. The argument reverses and results in the proof above.

(Hint for constructing the proof for $b > 1$: Can you relate $\bar{b}, b,$ and ε in such a way that you can still make $\frac{\bar{b}^n}{b^n} \leq \varepsilon$ for sufficiently large values of n?)