Five Interpolation Problem Solutions

1. Find a function $f:\mathbb{R}\to\mathbb{R}$ that is a linear combination of $1, e^x$, and $\sin x$ so that $f(0)=1$, $f(2)=1$, and $f(-1)=0$.

By letting $B=\begin{bmatrix}1 & e^0 & \sin(0)\\ 1 & e^2 & \sin(2)\\ 1 & e^{-1} & \sin(-1)\end{bmatrix}$ and $y=\begin{bmatrix}1\\1\\0\end{bmatrix}$, we find that

$$a=\frac{1}{(1-e^{-1})\sin(2)+(e^2-1)\sin(-1)}\begin{bmatrix}(e^2-1)\sin(-1)-e^{-1}\sin(2)\\ \sin(2)\\ 1-e^2\end{bmatrix} \text{ solves } Ba=y.$$

Thus the solution is $f(x)=\dfrac{(e^2-1)\sin(-1)-e^{-1}\sin(2)+\sin(2)e^x+(1-e^2)\sin(x)}{(1-e^{-1})\sin(2)+(e^2-1)\sin(-1)}$.

To about sixteen digits, the coefficients are $a=\begin{bmatrix}1.18938094937821\\ \text{-}0.18938094937821\\ 1.33065977528395\end{bmatrix}$.

2. Determine a Lagrangian basis for the space of functions spanned by 1 and x^2.

We seek to have $\tilde{g}^1, \tilde{g}^2 \in \langle 1, x^2\rangle$ with $\tilde{g}^1(x_1)=1, \tilde{g}^1(x_2)=0, \tilde{g}^2(x_1)=0$, and $\tilde{g}^2(x_2)=1$. By obtaining the inverse of the matrix $B=\begin{bmatrix}1 & x_1^2\\ 1 & x_2^2\end{bmatrix}$, we find

$\tilde{g}^1(x)=\dfrac{x^2-x_2^2}{x_1^2-x_2^2}$ and $\tilde{g}^2(x)=\dfrac{x^2-x_1^2}{x_2^2-x_1^2}$.

3. Determine a Lagrangian basis for the space of functions spanned by $1, x$, and e^x.

We seek to have $\tilde{g}^1, \tilde{g}^2, \tilde{g}^3 \in \langle 1, x, e^x\rangle$ with $\tilde{g}^1(x_1)=1, \tilde{g}^1(x_2)=0, \tilde{g}^1(x_3)=0$, $\tilde{g}^2(x_1)=0, \tilde{g}^2(x_2)=1, \tilde{g}^2(x_3)=0$, and $\tilde{g}^3(x_1)=0, \tilde{g}^3(x_2)=0, \tilde{g}^3(x_3)=1$. By obtaining the inverse of the matrix $B=\begin{bmatrix}1 & x_1 & e^{x_1}\\ 1 & x_2 & e^{x_2}\\ 1 & x_3 & e^{x_3}\end{bmatrix}$, we find

$$\tilde{g}^1(x)=\frac{x_2e^{x_3}-x_3e^{x_2}+(e^{x_2}-e^{x_3})x+(x_3-x_2)e^x}{x_1(e^{x_2}-e^{x_3})+x_2(e^{x_3}-e^{x_1})+x_3(e^{x_1}-e^{x_2})},$$

$$\tilde{g}^2(x)=\frac{x_3e^{x_1}-x_1e^{x_3}+(e^{x_3}-e^{x_1})x+(x_1-x_3)e^x}{x_1(e^{x_2}-e^{x_3})+x_2(e^{x_3}-e^{x_1})+x_3(e^{x_1}-e^{x_2})}, \text{ and}$$

$$\tilde{g}^3(x) = \frac{x_1 e^{x_2} - x_2 e^{x_1} + (e^{x_1} - e^{x_2})\, x + (x_2 - x_1) e^{x}}{x_1(e^{x_2} - e^{x_3}) + x_2(e^{x_3} - e^{x_1}) + x_3(e^{x_1} - e^{x_2})} \quad .$$

4. Consider the Planar Interpolation Problem:

Given $\{((x_1^1, x_1^2), y_1), ((x_2^1, x_2^2), y_2), ((x_3^1, x_3^2), y_3)\}$, find a function $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the form $p(x^1, x^2) = a_1 + a_2 x^1 + a_3 x^2$ so that for $i = 1,2,3$

$$p(x_i^1, x_i^2) = y_i .$$

Build a Lagrangian basis for the space of functions spanned by $1, x^1$, and x^2 .

We seek to have $\tilde{g}^1, \tilde{g}^2, \tilde{g}^3 \in \langle 1, x^1, x^2 \rangle$ with

$\tilde{g}^1(x_1^1, x_1^2) = 1, \tilde{g}^1(x_2^1, x_2^2) = 0, \tilde{g}^1(x_3^1, x_3^2) = 0,$

$\tilde{g}^2(x_1^1, x_1^2) = 0, \tilde{g}^2(x_2^1, x_2^2) = 1, \tilde{g}^2(x_3^1, x_3^2) = 0,$ and

$\tilde{g}^3(x_1^1, x_1^2) = 0, \tilde{g}^3(x_2^1, x_2^2) = 0, \tilde{g}^3(x_3^1, x_3^2) = 1$. By obtaining the inverse of the matrix

$B = \begin{bmatrix} 1 & x_1^1 & x_1^2 \\ 1 & x_2^1 & x_2^2 \\ 1 & x_3^1 & x_3^2 \end{bmatrix}$, we find

$$\tilde{g}^1(x^1, x^2) = \frac{x_2^1 x_3^2 - x_3^1 x_2^2 + (x_2^2 - x_3^2)x^1 + (x_3^1 - x_2^1)x^2}{x_1^1(x_2^2 - x_3^2) + x_2^1(x_3^2 - x_1^2) + x_3^1(x_1^2 - x_2^2)} \quad ,$$

$$\tilde{g}^2(x^1, x^2) = \frac{x_3^1 x_1^2 - x_1^1 x_3^2 + (x_3^2 - x_1^2)x^1 + (x_1^1 - x_3^1)x^2}{x_1^1(x_2^2 - x_3^2) + x_2^1(x_3^2 - x_1^2) + x_3^1(x_1^2 - x_2^2)} \quad \text{, and}$$

$$\tilde{g}^3(x^1, x^2) = \frac{x_1^1 x_2^2 - x_2^1 x_1^2 + (x_1^2 - x_2^2)x^1 + (x_2^1 - x_1^1)x^2}{x_1^1(x_2^2 - x_3^2) + x_2^1(x_3^2 - x_1^2) + x_3^1(x_1^2 - x_2^2)} \quad .$$

These can be slightly rearranged to be

$$\tilde{g}^1(x^1, x^2) = \frac{(x_2^2 - x_3^2)(x^1 - x_2^1) + (x_3^1 - x_2^1)(x^2 - x_2^2)}{(x_2^2 - x_3^2)(x_1^1 - x_2^1) + (x_3^1 - x_2^1)(x_1^2 - x_2^2)} \quad ,$$

$$\tilde{g}^2(x^1, x^2) = \frac{(x_3^2 - x_1^2)(x^1 - x_3^1) + (x_1^1 - x_3^1)(x^2 - x_3^2)}{(x_3^2 - x_1^2)(x_2^1 - x_3^1) + (x_1^1 - x_3^1)(x_2^2 - x_3^2)} \quad \text{, and}$$

$$\tilde{g}^3(x^1, x^2) = \frac{(x_1^2 - x_2^2)(x^1 - x_1^1) + (x_2^1 - x_1^1)(x^2 - x_1^2)}{(x_1^2 - x_2^2)(x_3^1 - x_1^1) + (x_2^1 - x_1^1)(x_3^2 - x_1^2)} \quad .$$

This form actually can be derived fairly directly – as opposed to using the inverse. It also requires significantly less arithmetic to evaluate.

5. Find a function $f:\mathbb{R}\to\mathbb{R}$ of the $f(x)=\dfrac{ax+b}{x+c}$ so that $f(0)=1$, $f(2)=-1$, and $f(-1)=0$.

We have the equations: $\dfrac{a\cdot 0+b}{0+c}=\dfrac{b}{c}=1, \dfrac{a\cdot 2+b}{2+c}=-1,$ and $\dfrac{a\cdot -1+b}{-1+c}=\dfrac{b-a}{c-1}=0.$ From the first and last equations we get $a=b=c$ and then from the second equation we have $a=b=c=-\dfrac{1}{2}$. The solution is then $f(x)=\dfrac{x+1}{1-2x}$.

Symbolic MATLAB Solutions

1.
```
x = [sym('0'); sym('2'); sym('-1')];
B = [ones(3,1) exp(x) sin(x)];
y = [1; 1; 0];
a = A\y

a_num = double(a)
```

2.
```
syms x1 x2
 x = [x1; x2];
B = [ones(2,1) x.^2];
G = inv(B)
```

3.
```
syms x1 x2 x3
x = [x1; x2; x3];
B = [ones(2,1) x exp(x)];
G = inv(B)
```

4.
```
syms x11 x12 x13 x21 x22 x23
x1 = [x11; x12; x13];
x2 = [x21; x22; x23];
B = [ones(3,1) x1 x2];
G = inv(B)
```

5.
```
syms a b c
s = solve('(a*0+b)/(0+c)=1','(a*2+b)/(2+c)=-1','(a*(-1)+b)/(-1+c)=0' );
s = [s.a, s.b, s.c]
```