Gaussian Quadrature Rules

Problem: *Given a domain of integration Ω and a $2n$ parameter class of real-valued functions G (where $n>0$) defined on Ω, determine $\{x_j\}_{j=1}^n$ and $\{w_j\}_{j=1}^n$ so that $\int_\Omega g = \sum_{j=1}^n w_j g(x_j)$, for all $g \in G$.*

For the rule to be usable, it would advantageous to have $x_j \in \Omega$ since the elements of G may not be defined for $x_j \notin \Omega$.

Nonlinear Case: If the parameterization of the elements of G is expressed as $g(\cdot; a_1, a_2, ..., a_{2n})$, then the equations specifying $\{x_j\}_{j=1}^n$ and $\{w_j\}_{j=1}^n$ are

$$\int_\Omega g(\cdot; a_1, a_2, ..., a_{2n}) = \sum_{j=1}^n w_j g(x_j; a_1, a_2, ..., a_{2n}),$$

for all $a_1, a_2, ..., a_{2n}$.

Linear Case: If G is expressed by a basis as $\langle g_1, g_2, ..., g_{2n} \rangle$, then the equations specifying $\{x_j\}_{j=1}^n$ and $\{w_j\}_{j=1}^n$ are

$$\int_\Omega g_i = \sum_{j=1}^n w_j \, g_i(x_j).$$

Notice that even in the linear case, the unknowns $\{x_j\}_{j=1}^n$ enter the equations (in general) in a nonlinear fashion, so this does *not* result in a system of $2n$ linear equations in $2n$ unknowns.

However, the $\{w_j\}_{j=1}^n$ do enter in linear fashion, thus, if the $\{x_j\}_{j=1}^n$ could somehow be determined, then purely linear equations could be used to obtain $\{w_j\}_{j=1}^n$. This is what is key to what follows. In the case, where Ω is an interval of the real line and G is a space of "trended" polynomials of degree $2n-1$, we can determine first the $\{x_j\}_{j=1}^n$ and then $\{w_j\}_{j=1}^n$ in a fairly straightforward fashion.

To that end, we first consider the notion of a trended polynomial space: Given an interval $[a,b]$ and a function v positive on (a,b), the space of degree n polynomials with the trend v is

$$H_n^v = \{v \; p | \; p \, \text{isapolynomialofdegree} \, n\}.$$

Thus, H^{v}_{2n-1} is a linear space of dimension $2n.$. We will see that these three steps determine a Gaussian quadrature rule on $[a,b]$ for H^{v}_{2n-1}.

1. Find $\tilde{g}_n$, a polynomial of degree n orthogonal to all polynomials of lesser degree with respect to the inner product $(f,h)=\int_a^b v(x)f(x)h(x)dx$.

2. Find the zeros of $\tilde{g}_n$ (there will be n distinct zeros on (a,b)) and set $\{x_i\}_{i=1}^n$ to these zeros.

3. Solve the linear system $\int_\Omega g_i = \sum_{j=1}^n w_j\, g_i(x_j)$ using some basis set $\langle g_1, g_2,..., g_n\rangle$ for H^{v}_{n}.

The polynomial $\tilde{g}_n$ can be found by applying the Gram-Schmidt algorithm to $1, x, x^2, ...x^n$. (There is also a technique for determining $\tilde{g}_n$ by developing a three term recurrence relation - see http://en.wikipedia.org/wiki/Orthogonal_polynomials#Recurrence_relations.)

The following lemma guarantees that the quadrature rule will be applicable to functions that may not be defined outside the domain of integration $[a,b]$.

Lemma: *The zeros of* $\tilde{g}_n$ *are simple and contained in* (a,b).

Proof:

With appropriate renumbering let $\{x_j\}_{j=1}^t$ be the distinct odd-order zeros of $\tilde{g}_n$ contained in (a,b). Since the total number of zeros cannot exceed n, and the number of zeros that are of odd-order and contained in (a,b) must be less than or equal to that, we have $t \le n$. Suppose $t < n$. Consider the t^{th} degree polynomial $p(x)=(x-x_1)(x-x_2)\cdots(x-x_3)$. The zeros of the product $p\cdot\tilde{g}_n$ that are in (a,b) must have even order. Thus $p\cdot\tilde{g}_n$ does not change sign on (a,b). Also $v\cdot p\cdot\tilde{g}_n$ does not change sign on (a,b) and, having only a finite number of zeros, cannot have a zero integral. But by orthogonality, since p has degree less than n, $\int_a^b v(x)p(x)\tilde{g}_n(x)dx=0$. This is a contradiction so $t=n$. We conclude that there are n odd-ordered zeros of $\tilde{g}_n$ contained in (a,b). The total multiplicity of all zeros (including in the complex plane) equals n, yet there are n distinct odd-ordered zeros of $\tilde{g}_n$ contained in (a,b). This precludes any multiple zeros. We conclude that all zeros of $\tilde{g}_n$ are simple and contained in (a,b). ð

Next we see that the linear system for the set $\{w_j\}_{j=1}^{n}$ has a unique solution. We know that the polynomial interpolation problem has a unique solution so the matrix with i, j component x_i^{j-1} for $i, j = 1,...,n$ is nonsingular. Its transpose (the matrix with i, j component x_j^{i-1} for $i, j = 1,...,n$) must also be nonsingular. Finally, the matrix with i, j component $v(x_j)x_j^{i-1}$ for $i, j = 1,...,n$ is simply the previous matrix post-multiplied by the non-singular diagonal matrix with positive j, j elements $v(x_j)$. Thus, it too is non-singular, and thus, by using the basis $\langle 1, x,..., x^{n-1} \rangle$ for polynomials of degree $n-1$, the linear system for the set $\{w_j\}_{j=1}^{n}$ has a unique solution. The solution is basis independent, however, so if there is a unique solution with one basis there is a unique solution for any basis.

Lastly we see that the sets $\{w_j\}_{j=1}^{n}$ and $\{x_j\}_{j=1}^{n}$ provide a quadrature rule for H_{2n-1}^{v}.

Theorem: *For the sets* $\{w_j\}_{j=1}^{n}$ *and* $\{x_j\}_{j=1}^{n}$ *so determined,* $\int_a^b g = \sum_{i=1}^{n} w_j g(x_j)$ *for all* $g \in H_{2n-1}^{v}$.

Proof:

Any $g \in H_{2n-1}^{v}$ must have the form $g = v \cdot p$, where p is of degree $2n-1$. Divide p by $\tilde{g}_n$ to get quotient s and remainder r. That is: $p = s \cdot \tilde{g}_n + r$. Both s and r must have degree at most $n-1$. We have

$$\int_a^b g = \int_a^b v \cdot p = \int_a^b v \cdot (s\tilde{g}_n + r) = \int_a^b v \cdot s\tilde{g}_n + \int_a^b v \cdot r$$

But by orthogonality the first expression is zero, and thus

$$\int_a^b g = \int_a^b v \cdot r,$$

and since r has degree at most $n-1$

$$\int_a^b g = \sum_{i=1}^{n} w_j v(x_j) r(x_j).$$

Since every x_j is a zero of $\tilde{g}_n$, $v(x_j) r(x_j) = v(x_j)\,(s(x_j) g_n(x_j) + r(x_j)) = g(x_j)$, so

$$\int_a^b g = \sum_{i=1}^{n} w_j g(x_j)\ .ð$$