Lemma: Given $a < b$, a function g with n continuous derivatives on $[a, b]$ and $n+1$ zeros on $[a, b]$ then there exists a point $\xi \in [a,b]$ so that $g^{(n)}(\xi) = 0$.

Proof:

We shall prove that if a function g on $[a, b]$ has m continuous derivatives and p zeros, then the derivative g' has $m-1$ continuous derivatives and $p-1$ zeros on $[a, b$. Inductively this proves the lemma. Let the zeros of g be $w_1 < w_2 < \cdots < w_p$. On each open interval (w_i, w_{i+1}) g' must have a zero by Rolle's Theorem. That is a total of $p-1$ distinct zeros and they all lie in $[a, b]$. Inductively we have then that $g^{(n)}$ has at least on zero in $[a, b]$.

Theorem: Given $a < b$, a function h with n continuous derivatives on $[a, b]$, a polynomial f of degree $n-1$ so that

$$f(x_i) = h(x_i) \text{ for } i = 1, \ldots, n ,$$

where the set $x_i \in [a,b]$ and are distinct, then for $x \in [a,b]$ there exists a point $\xi \in [a,b]$ so that

$$h(x) - f(x) = \frac{(x - x_1)\cdots(x - x_n)}{n!} h^{(n)}(\xi)$$

Proof:

Consider any $x \in [a,b]$. If $x = x_i$ for some i then the result clearly holds since both sides are zero. Now suppose x is distinct from any of the x_i and consider the function

$$l(z) = h(z) - f(z) - \frac{(h(x) - f(x))(z - x_1)\cdots(z - x_n)}{(x - x_1)\cdots(x - x_n)}.$$

Since the difference $h(z) - f(z)$ has a zero at x_i and so does the term $(z - x_i)$, thus l itself has a zero at x_i. Lastly, it's easy to see that $l(x) = 0$. We then have that l has $n+1$ zeroes on $[a,b]$. Since h has n continuous derivatives so does l. The lemma can then be applied to guarantee there exists a point $\xi \in [a,b]$ so that $l^{(n)}(\xi) = 0$. But since f is of degree $n-1$, its nth derivative is everywhere zero. Also, the product $(z - x_1)\cdots(z - x_n)$, if expanded, would be of the form $z^n + \overline{p}$ for some polynomial $\overline{p}$ of degree $n-1$. The nth derivative of this product is the constant $n!$. Thus,

$$0 = l^{(n)}(\xi) = h^{(n)}(\xi) - 0 - \frac{(h(x) - f(x))n!}{(x - x_1)\cdots(x - x_n)}$$

By rearranging terms we obtain

$$h(x) - f(x) = \frac{(x - x_1)\cdots(x - x_n)}{n!} h^{(n)}(\xi)$$