Orthogonality Property on Inner Product Spaces

Lemma: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$ *, then* $\|x-y\|^2 = \|x\|^2 - 2(x,y) + \|y\|^2$ *for any elements* $x, y \in S$.

Proof:

We have

$$\begin{aligned}\|x-y\|^2 &= (x-y, x-y)\\ &= (x,x) - (x,y) - (y,x) + (y,y)\\ &= \|x\|^2 - 2(x,y) + \|y\|^2\end{aligned}$$

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Theorem: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. An element* g^* *of* G *minimizes* $\|f-g\|$ *over all elements* $g \in G$ *if and only if* $(f-g^*, g) = 0$ *for every* $g \in G$.

Proof:

First we will prove that if $(f-g^*, g) = 0$ for every $g \in G$ then g^* of G minimizes $\|f-g\|$ over all elements $g \in G$. Consider any $\tilde{g} \in G$, then

$$\begin{aligned}\|f-\tilde{g}\|^2 &= \|f-g^*-(\tilde{g}-g^*)\|^2\\ &= \|f-g^*\|^2 - 2(f-g^*, \tilde{g}-g^*) + \|\tilde{g}-g^*\|^2\end{aligned}$$

Since both $\tilde{g}$ and g^* are elements of G, so is $\tilde{g}-g^*$, and by assumption $(f-g^*, \tilde{g}-g^*) = 0$. Furthermore, $\|\tilde{g}-g^*\|^2 \geq 0$, so we conclude that

$$\|f-\tilde{g}\|^2 \geq \|f-g^*\|^2.$$

Now we will prove that if $(f-g^*, \hat{g}) \neq 0$ for some $\hat{g} \in G$ then g^* of G cannot minimize $\|f-g\|$ over all elements $g \in G$. Without loss of generality, we assume $(f-g^*, \hat{g}) > 0$. (If $(f-g^*, \hat{g}) < 0$ then, since $(f-g^*, -\hat{g}) > 0$, the argument works with $-\hat{g}$ instead of $\hat{g}$.) Now consider the element $g^* + t\hat{g}$ of G for real values of t. We have

$$\begin{aligned}\|f-(g^*+t\hat{g})\|^2 &= \|f-g^*-t\hat{g}\|^2\\ &= \|f-g^*\|^2 - 2(f-g^*, t\hat{g}) + \|t\hat{g}\|^2\\ &= \|f-g^*\|^2 - 2t(f-g^*, \hat{g}) + t^2\|\hat{g}\|^2\end{aligned}$$

The quantity $\frac{2(f-g^*,\hat{g})}{\|\hat{g}\|^2}$ is necessarily positive. Select a positive $t < \frac{2(f-g^*,\hat{g})}{\|\hat{g}\|^2}$. This implies that $t\|\hat{g}\|^2 < 2(f-g^*,\hat{g})$ and $t^2\|\hat{g}\|^2 < 2t(f-g^*,\hat{g})$ so

$-2t(f-g^*,\hat{g}) + t^2\|\hat{g}\|^2 < 0$ and

$$\begin{aligned}\|f-(g^*+t\hat{g})\|^2 &= \|f-g^*\|^2 - 2t(f-g^*,\hat{g}) + t^2\|\hat{g}\|^2 \\ &< \|f-g^*\|^2\end{aligned}$$

We conclude that g^* cannot minimize $\|f-g\|$ over all elements $g \in G$. ð

Corollary: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. There exists a unique element* g^* *of* G *that minimizes* $\|f-g\|$.

Proof:

We have previously proved that there exists some element g^* of G that minimizes $\|f-g\|$ over all elements $g \in G$. Suppose there is distinct element $\tilde{g} \in G$ so that $\|f-\tilde{g}\| = \|f-g^*\|$. From the theorem above, we know that $\|f-\tilde{g}\|^2 = \|f-g^*\|^2 + \|\tilde{g}-g^*\|^2$ but since $\tilde{g} \neq g^*$, we have $\|\tilde{g}-g^*\|^2 > 0$, so $\|f-\tilde{g}\|^2 > \|f-g^*\|^2$ contrary to assumption. We conclude that there is a unique minimizer.