Orthogonality Property on Inner Product Spaces

**Lemma:** Let $S$ be an inner product space with norm $\|\|$ induced by the inner product $(\cdot, \cdot)$, then
\[
\|x - y\|^2 = \|x\|^2 - 2(x, y) + \|y\|^2
\]
for any elements $x, y \in S$.

**Proof:**
We have
\[
\|x - y\|^2 = (x - y, x - y) \\
= (x, x) - (x, y) - (y, y) + (y, y) \\
= \|x\|^2 - 2(x, y) + \|y\|^2
\]

**Theorem:** Let $S$ be an inner product space with norm $\|\|$ induced by the inner product $(\cdot, \cdot)$, $G$ be a finite dimensional subspace of $S$, and $f$ an element of $S$. An element $g^*$ of $G$ minimizes $\|f - g\|$ over all elements $g \in G$ if and only if $(f - g^*, g) = 0$ for every $g \in G$.

**Proof:**
First we will prove that if $(f - g^*, g) = 0$ for every $g \in G$ then $g^*$ of $G$ minimizes $\|f - g\|$ over all elements $g \in G$. Consider any $\tilde{g} \in G$, then
\[
\|f - \tilde{g}\|^2 = \|f - g^* - (\tilde{g} - g^*)\|^2 \\
= \|f - g^*\|^2 - 2(f - g^*, \tilde{g} - g^*) + \|\tilde{g} - g^*\|^2
\]
Since both $\tilde{g}$ and $g^*$ are elements of $G$, so is $\tilde{g} - g^*$, and by assumption $(f - g^*, \tilde{g} - g^*) = 0$. Furthermore, $\|\tilde{g} - g^*\|^2 \geq 0$, so we conclude that
\[
\|f - \tilde{g}\|^2 \geq \|f - g^*\|^2.
\]

Now we will prove that if $(f - g^*, \tilde{g}) \neq 0$ for some $\tilde{g} \in G$ then $g^*$ of $G$ cannot minimize $\|f - g\|$ over all elements $g \in G$. Without loss of generality, we assume $(f - g^*, \tilde{g}) > 0$. (If $(f - g^*, \tilde{g}) < 0$ then, since $(f - g^*, -\tilde{g}) > 0$, the argument works with $-\tilde{g}$ instead of $\tilde{g}$.) Now consider the element $g^* + t\tilde{g}$ of $G$ for real values of $t$. We have
\[
\|f - (g^* + t\tilde{g})\|^2 = \|f - g^* - t\tilde{g}\|^2 \\
= \|f - g^*\|^2 - 2(f - g^*, t\tilde{g}) + \|t\tilde{g}\|^2 \\
= \|f - g^*\|^2 - 2t(f - g^*, \tilde{g}) + t^2 \|\tilde{g}\|^2
\]
The quantity \( \frac{2(f - g^* \cdot \bar{g})}{\|\bar{g}\|^2} \) is necessarily positive. Select a positive \( t < \frac{2(f - g^* \cdot \bar{g})}{\|\bar{g}\|^2} \). This implies that \( t\|\bar{g}\|^2 < 2(f - g^* \cdot \bar{g}) \) and \( r^2\|\bar{g}\|^2 < 2t(f - g^* \cdot \bar{g}) \) so

\[-2t(f - g^* \cdot \bar{g}) + r^2\|\bar{g}\|^2 < 0 \quad \text{and} \]

\[\left\| f - (g^* + t\bar{g}) \right\|^2 = \left\| f - g^* \right\|^2 - 2t(f - g^* \cdot \bar{g}) + r^2\|\bar{g}\|^2 < \left\| f - g^* \right\|^2\]

We conclude that \( g^* \) cannot minimize \( f \cdot g \) over all elements \( g \in G \).

**Corollary:** Let \( S \) be an inner product space with norm \( \| \| \) induced by the inner product \( (.,.) \), \( G \) be a finite dimensional subspace of \( S \), and \( f \) an element of \( S \). There exists a unique element \( g^* \) of \( G \) that minimizes \( \| f - g \| \).

**Proof:**

We have previously proved that there exists some element \( g^* \) of \( G \) that minimizes \( \| f - g \| \) over all elements \( g \in G \). Suppose there is distinct element \( \bar{g} \in G \) so that

\[\| f - \bar{g} \| = \| f - g^* \| \]. From the theorem above, we know that

\[\| f - \bar{g} \|^2 = \| f - g^* \|^2 + \| \bar{g} - g^* \|^2 \] but since \( \bar{g} \neq g^* \), we have \( \| \bar{g} - g^* \|^2 > 0 \), so

\[\| f - \bar{g} \|^2 > \| f - g^* \|^2\] contrary to assumption. We conclude that there is a unique minimizer.