Four Corollaries to the Orthogonality Property on Inner Product Spaces

Theorem: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. An element* g^* *of* G *minimizes* $\|f-g\|$ *over all elements* $g \in G$ *if and only if* $(f-g^*, g) = 0$ *for every* $g \in G$.

Corollary 1: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. An element* g^* *of* G *minimizes* $\|f-g\|$ *over all elements* $g \in G$ *if and only if* $(f, g) = (g^*, g)$ *for every* $g \in G$.

> **Proof:** From the linearity of the inner product, we have $(f-g^*, g) = 0$ if and only if $(f, g) = (g^*, g)$.

Corollary 2: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. Let* $< g_1, g_2, ..., g_n >$ *be a basis for* G. *An element* g^* *of* G *minimizes* $\|f-g\|$ *over all elements* $g \in G$ *if and only if* $(f, g_i) = (g^*, g_i)$ *for* $i = 1,2,...,n.$

> **Proof:** Obviously, if $(f, g) = (g^*, g)$ for every $g \in G$ then $(f, g_i) = (g^*, g_i)$ for $i = 1,2,...,n.$ Now assume $(f, g_i) = (g^*, g_i)$ for $i = 1,2,...,n,$ where $< g_1, g_2, ..., g_n >$ is a basis for $G.$ Any $g \in G$ can be written as a linear combination of the basis elements: $g = \sum_{j=1}^{n} a_j g_j$, but then we have:
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> $$(f, g) = (f, \sum_{j=1}^{n} a_j g_j) = \sum_{j=1}^{n} a_j (f, g_j) = \sum_{j=1}^{n} a_j (g^*, g_j) = (g^*, \sum_{j=1}^{n} a_j g_j) = (g^*, g).$$

Corollary 3: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. Let* $< g_1, g_2, ..., g_n >$ *be a basis for* G. *An element* g^* *of* G *with expansion* $\sum_{j=1}^{n} a_j^* g_j$ *minimizes* $\|f-g\|$ *over all elements* $g \in G$ *if and only if*

$$\sum_{j=1}^{n} a_j^* (g_j, g_i) = (f, g_i) \text{ for } i = 1,2,...,n.$$

> **Proof:** Since $g^* = \sum_{j=1}^{n} a_j^* g_j$, for $i = 1,...,n,$ by linearity of the inner product then condition $(g^*, g_i) = (f, g_i)$ is equivalent to $\sum_{j=1}^{n} a_j^* (g_j, g_i) = (f, g_i)$.

Definition: A set $\{\tilde{g}_1, \tilde{g}_2, ..., \tilde{g}_n\}$ of elements of an inner product space is *orthogonal* if and only if for $i, j = 1, ..., n$ and $i \neq j$ $(\tilde{g}_i, \tilde{g}_j) = 0$. A set $\{\tilde{g}_1, \tilde{g}_2, ..., \tilde{g}_n\}$ of elements of an inner product space is *orthonormal* if and only if it is orthogonal and each element has unit norm.

Corollary 4: *Let S be an inner product space with norm* $\|.\|$ *induced by the inner product* $(.,.)$, G *be a finite dimensional subspace of S, and f an element of S. Let* $< \tilde{g}_1, \tilde{g}_2, ..., \tilde{g}_n >$ *be an orthonormal basis for* G. *The element* $g^* = \sum_{j=1}^{n} (f, \tilde{g}_j)\tilde{g}_j$ *of* G *minimizes* $\|f - g\|$ *over all elements* $g \in G$.

Proof: From Corollary 3 we have that g^*, the minimizer of $\|f - g\|$ over G, with expansion satisfies $\sum_{j=1}^{n} a_j^*(\tilde{g}_j, \tilde{g}_i) = (f, \tilde{g}_i)$ for $i = 1,2,...,n$. But if $< \tilde{g}_1, \tilde{g}_2, ..., \tilde{g}_n >$ is an orthonormal basis for G, then $\sum_{j=1}^{n} a_j^*(\tilde{g}_j, \tilde{g}_i) = a_i^*(\tilde{g}_i, \tilde{g}_i) = a_i^* = (f, \tilde{g}_i)$.

The expression $g^* = \sum_{j=1}^{n} (f, \tilde{g}_j)\tilde{g}_j$ in terms of an orthonormal basis is sometimes termed a *Fourier expansion* and the coefficients $(f, \tilde{g}_j)$ termed *Fourier coefficients*. If the orthogonal basis is not necessarily orthonormal, the Fourier expansion takes the form

$$g^* = \sum_{j=1}^{n} \frac{(f, \tilde{g}_j)}{(\tilde{g}_j, \tilde{g}_j)} \tilde{g}_j \ .$$