M 340L – CS
Homework Set 8

1. Let $u^1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u^2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, u^3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

a. Form the matrix $U = \begin{bmatrix} u^1 & u^2 & u^3 \end{bmatrix}$ and confirm that the columns of U are orthogonal by computing $U^T U$.

b. Express b as a linear combination of u^1, u^2 and u^3. (That is, solve $Ux = b$. Be clever about using U^T to do this.)

2. Let $u^1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, u^2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

a. Form the matrix $U = \begin{bmatrix} u^1 & u^2 \end{bmatrix}$ and confirm that the columns of U are orthogonal by computing $U^T U$.

b. Normalize the columns and confirm that $U^T U = I$.

3. Answer true or false to the following. If false offer a simple counterexample.

a. Every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{u^1, u^2, \ldots, u^k\}$ has the property that $u^i \cdot u^j = 0$ whenever $i \neq j$, then S is an orthonormal set.

4. Show that if U is a square matrix whose columns are orthonormal then $U^T = U^{-1}$.

5. Show that if U is an $m \times n$ orthogonal matrix then for all $x \in \mathbb{R}^n, \|Ux\| = \|x\|$. (This is not hard: work out $\|Ux\|^2$. This can be stated as "An orthogonal transformation preserves length.".)

6. Consider this mathematical (and not necessarily computer) procedure:

$[\alpha, v'] =$ **project** $[u, v]$

Inputs vectors u and v, computes and returns $\alpha = u \cdot v / u \cdot u$ and $v' = v - \alpha u$.

Now, let $u^1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u^2 = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}, u^3 = \begin{bmatrix} 9 \\ -3 \\ 3 \end{bmatrix}$.

a. Perform $[r_{1,2}, u_2'] =$ **project** $[u_1, u_2]$.(That is, subtract the projection of u_2 onto the subspace spanned by u_1.)

b. Perform $[r_{1,3}, u_3'] =$ **project** $[u_1, u_3]$.(That is, subtract the projection of u_3 onto the subspace spanned by u_1.)

c. Perform $[r_{2,3}, u_3''] =$ **project** $[u_2', u_3']$.(That is, subtract the projection of u_3' onto the subspace spanned by u_1.)

d. Compute $A = \begin{bmatrix} u_1 & u_2' & u_3'' \end{bmatrix} \begin{bmatrix} 1 & r_{1,2} & r_{1,3} \\ 0 & 1 & r_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$. (Compare to U in Problem 1. You have just used the Gram-Schmidt Algorithm to orthogonalize – but not ortho**normalize** - vectors. That is, the normalizations are not done.)

7. Prove that if $y^T x = 0$, for all x, then $y = 0$. (Hint: Consider $x = y$, in particular.)

8. Prove that if $Q^T Q = I$, then if x is perpendicular to y, then Qx is perpendicular to Qy.