Exam 2 Practice Problems

Theory:

T1. Let u and v be any n-vectors so that $v^T u = 1$. Define $P = I - uv^T$ (note that this involves the outer product uv^T, which is a matrix, and not the inner product $u^T v$, which is a scalar). Prove that P is a projection by showing $P^2 = P$. (Hint: Consider associativity and look for $v^T u$. Have a string of equalities with P^2 at one end and P at the other.)

T2.. Using the definition of a projection P (i.e., that $P^2 = P$), prove that if P is also invertible, it must be the identity.

T3. Consider the least squares problem:

$$\text{Determine } x^* \in \mathbb{R}^n, \text{ so that } \|Ax^* - b\| \leq \|Ax - b\| \text{ for all } x \in \mathbb{R}^n.$$

We know if the matrix A has linearly independent columns then x^* can be determined by solving the normal equations

$$A^T A x^* = A^T b.$$

Prove that if x^* satisfies the normal equations then it also satisfies

$$Rx^* = Q^T b,$$

where $A = QR$, Q satisfies $Q^T Q = I$, and R is invertible.

T4. Given an orthogonal matrix U, prove that

a. For all vectors $x, \|x\| = \|Ux\|$.
b. For all vectors x and $y, x \cdot y = (Ux) \cdot (Uy)$.
c. For all vectors x and y, if x is perpendicular to y then Ux is perpendicular to Uy.

Gram-Schmidt

GS1. Given $A = \begin{bmatrix} 2 & -6 \\ 1 & 3 \\ 2 & 0 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR,$ where $Q^T Q = I$ and R is upper triangular.

GS2. Given $A = \begin{bmatrix} 3 & -5 \\ 1 & 1 \\ -1 & 5 \\ 3 & -7 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR,$, where $Q^T Q = I$ and R is upper triangular.

GS3. Given $A = \begin{bmatrix} -2 & 4 \\ 1 & 4 \\ 0 & 2 \\ 2 & -7 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR,$, where $Q^T Q = I$ and R is upper triangular.

GS4. Suppose that applying the Gram-Schmidt algorithm to $A = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}$ results in $A = QR,$

where $\underset{\sim}{Q} = \begin{bmatrix} -1/2 & 1/\sqrt{2} \\ 1/2 & 0 \\ 1/2 & 1/\sqrt{2} \\ 1/2 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 4 & -2 \\ 0 & \sqrt{2} \end{bmatrix}$. (The matrix $\underset{\sim}{Q}$ satisfies $\underset{\sim}{Q}^T \underset{\sim}{Q} = I$.) Determine

$x*$ that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^2$, where $b = \begin{bmatrix} -4 \\ 4 \\ 2 \\ 2 \end{bmatrix}$.

Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.

GS5. [15] Suppose that applying the Gram-Schmidt algorithm to $A = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & 0 \end{bmatrix}$ results in

$A = QR,$ where $Q = \begin{bmatrix} -2/3 & 1/\sqrt{5} \\ 1/3 & 2/\sqrt{5} \\ 2/3 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5}/3 \end{bmatrix}$. (The matrix Q satisfies $Q^T Q = I$.)

Determine x^* that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^2$, where $b = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$.

Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.

True or False

TF1. Answer true or false:

________a. If A is a $n \times n$ matrix, then $(A^2)^T = (A^T)^2$.

________b. For A, an $n \times n$ matrix, t even if the equation $Ax = b$ has more than one solution (for some $Ax = b$), the transformation $x \mapsto Ax$ can be one-to-one.

________**c.** If the columns of an $n \times n$ matrix are linearly independent then the columns must span $\mathbb{R}^n$.

________**d.** If the matrices A and B are invertible, then the matrix $A + B$ is invertable.

________**e.** If A is a $n \times n$ matrix, then $(A^2)^T = (A^T)^2$.

________**f.** For A, an $n \times n$ matrix, t even if the equation $Ax = b$ has more than one solution (for some $Ax = b$), the transformation $x \mapsto Ax$ can be one-to-one.

________**g.** If the columns of an $n \times n$ matrix are linearly independent then the columns must span $\mathbb{R}^n$.

________**h.** If the matrices A and B are invertible, then the matrix $A + B$ is invertable.

________**i.** If the problem $Ax = b$ has any solution x, then the null space of must be only $\{0\}$.

________**j.** If the problem $Ax = b$ has any solution x, then b must be in the column space of A.

________**k.** For any $m \times n$ matrix A, the null space of A is a subspace of $\mathbb{R}^m$.

________**l.** For any square matrix A, if the rows are linearly independent so are the columns.

Vector Spaces

VS1. Identify which of the following satisfy the definition for vector spaces. For each case mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

	Closed under addition	Closed under scalar multiplication
a. The set of four by four matrices A such that $A_{1,1} = 0$.	________	________
b. The set of polynomials of degree at most ten	________	________
c. The set of ordered triples of real numbers (a,b,c) so that $\|a-1\| \le 1, \|b-1\| \le 1$, and $\|c-1\| \le 1$.	________	________
d. The set of ordered pairs of real numbers (a,b) so that $ab \le 0$.	________	________
e. The set of infinite arrays $[x_1, x_2, x_3, \ldots]$ all of whose elements are non-negative	________	________
f. The set of three by three upper triangular matrices.	________	________
g. The set of functions $f:[0,1] \to \mathbb{R}$	________	________
h. The set of ordered pairs of real numbers (a,b) so that $a \ge b$.	________	________
i. The set of ordered triples of real numbers (a,b,c) so that $b = 1$.	________	________
j. The set of two by two invertible matrices.	________	________

Null Spaces, Inverses, etc.

N1. a. Find the null space of the matrix $A = \begin{bmatrix} 4 & 2 & -3 \\ 2 & 1 & 5 \\ 2 & 1 & -2 \end{bmatrix}$.

b. For the matrix A in part a. find a solution to $Ax = \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix}$ or show no solution exists.

N2. Display a matrix A so that $A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. (Hint: If the solution is not obvious, determine some equations for the elements of A and solve them.)

N3 a. Find the null space of the matrix $A = \begin{bmatrix} 2 & 8 & 4 \\ 1 & 4 & 4 \\ -1 & -4 & -3 \end{bmatrix}$.

b. For the matrix A in part a. find a solution to $Ax = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ or show no solution exists.