## Exam 2 Practice Problems

**Theory:**

**T1. a**. Let $u$ and $v$ be any $n$-vectors so that $v^T u = 1$. Define $P = I - uv^T$ (note that this involves the outer product $uv^T$, which is a matrix, and not the inner product $u^T v$, which is a scalar). Prove that $P$ is a projection by showing $P^2 = P$. (Hint: Consider associativity and look for $v^T u$. Have a string of equalities with $P^2$ at one end and $P$ at the other.)

We have:

$$\begin{aligned} P^2 &= (I - uv^T)(I - uv^T) \\ &= I - 2uv^T + uv^T uv^T \\ &= I - 2uv^T + u(v^T u)v^T \\ &= I - 2uv^T + uv^T \\ &= I - uv^T \\ &= P. \end{aligned}$$

**T2.**. Using the definition of a projection $P$ (i.e., that $P^2 = P$), prove that if $P$ is also invertible, it must be the identity.

Since $P$ is a projection $P^2 = P,$ and since $P$ is invertible $P = P^{-1}P^2 = P^{-1}P = I.$ thus $P$ must be the identity.

**T3.** Consider the least squares problem:

$$\text{Determine } x^* \in \mathbb{R}^n, \text{ so that } \|Ax^* - b\| \leq \|Ax - b\| \text{ for all } x \in \mathbb{R}^n.$$

We know if the matrix $A$ has linearly independent columns then $x^*$ can be determined by solving the normal equations

$$A^T A x^* = A^T b.$$

Prove that if $x^*$ satisfies the normal equations then it also satisfies

$$Rx^* = Q^T b,$$

where $A = QR,$ $Q$ satisfies $Q^T Q = I,$ and $R$ is invertible.

We have $A^T A x^* = A^T b$, so $R^T R x^* = R^T Q^T Q R x^* = A^T A x^* = A^T b = (QR)^T b = R^T Q^T b.$ Since $R$ is invertible so is $R^T$. Thus, we have

$$Rx^* = (R^T)^{-1} R^T R x^* = (R^T)^{-1} R^T Q^T b = Q^T b.$$

**T4**. Given an orthogonal matrix U, prove that

**a.** For all vectors $x, \|x\| = \|Ux\|$.

Since $U^T U = I$, we have for all vectors $x, \|x\|^2 = x^T x = x^T Ix = x^T U^T Ux = (Ux)^T (Ux) = \|Ux\|^2$.

**b.** For all vectors $x$ and $y, x \cdot y = (Ux) \cdot (Uy)$.

Since $U^T U = I$, we have for all vectors $x$ and $y$,
$x \cdot y = x^T y = x^T Iy = x^T U^T Uy = (Ux)^T (Uy) = (Ux) \cdot (Uy)$.

**c.** For all all vectors $x$ and $y$, if $x$ is perpendicular to $y$ then $Ux$ is perpendicular to $Uy$.

Since $U^T U = I$, we have for all vectors $x$ and $y$, assume $x \cdot y = 0$, then
$0 = x \cdot y = x^T y = x^T Iy = x^T U^T Uy = (Ux)^T (Uy) = (Ux) \cdot (Uy)$. So if $x$ is perpendicular to $y$ then $Ux$ is perpendicular to $Uy$.

**Gram-Schmidt**

**GS1.** Given $A = \begin{bmatrix} 2 & -6 \\ 1 & 3 \\ 2 & 0 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR$, where $Q^T Q = I$ and $R$ is upper triangular.

$$R_{1,1} = \left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\| = 3,\ Q_{.,1} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, R_{1,2} = Q_{.,1}^T A_{.,2} = -3,\ \bar{Q}_{.,2} = A_{.,2} - R_{1,2} Q_{.,1} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix},$$

$$R_{2,2} = \left\| \bar{Q}_{.,2} \right\| = 6, Q_{.,2} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}. \text{Thus, } A = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}.$$

**GS2.** Given $A = \begin{bmatrix} 3 & -5 \\ 1 & 1 \\ -1 & 5 \\ 3 & -7 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR,$, where $Q^T Q = I$ and $R$ is upper triangular.

$$R_{1,1} = \left\| \begin{matrix} 3 \\ 1 \\ -1 \\ 3 \end{matrix} \right\| = \sqrt{20}, Q_{.,1} = \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, R_{1,2} = Q_{.,1}^T A_{.,2} = -2/\sqrt{20},\ \bar{Q}_{.,2} = A_{.,2} - R_{1,2} Q_{.,1} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix},$$

$$R_{2,2} = \left\| \bar{Q}_{.,2} \right\| = \sqrt{20}, Q_{.,2} = \begin{bmatrix} 1/\sqrt{20} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}. \text{Thus, } A = \begin{bmatrix} 3/\sqrt{20} & 1/\sqrt{20} \\ 1/\sqrt{20} & 3/\sqrt{20} \\ -1/\sqrt{20} & 3/\sqrt{20} \\ 3/\sqrt{20} & -1/\sqrt{20} \end{bmatrix} \begin{bmatrix} \sqrt{20} & -2\sqrt{20} \\ 0 & \sqrt{20} \end{bmatrix}$$

**GS3.** Given $A = \begin{bmatrix} -2 & 4 \\ 1 & 4 \\ 0 & 2 \\ 2 & -7 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR$, where $Q^T Q = I$ and $R$ is upper triangular.

$$R_{1,1} = \left\| \begin{matrix} -2 \\ 1 \\ 0 \\ 2 \end{matrix} \right\| = 3,\ Q_{.,1} = \begin{bmatrix} -2/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix}, R_{1,2} = Q_{.,1}^T A_{.,2} = -6,\ \bar{Q}_{.,2} = A_{.,2} - R_{1,2} Q_{.,1} = \begin{bmatrix} 0 \\ 6 \\ 2 \\ -3 \end{bmatrix},$$

$$R_{2,2} = \left\| \bar{Q}_{.,2} \right\| = 7, Q_{.,2} = \begin{bmatrix} 0 \\ 6/7 \\ 2/7 \\ -3/7 \end{bmatrix}. \text{Thus, } A = \begin{bmatrix} -2/3 & 0 \\ 1/3 & 6/7 \\ 0 & 2/7 \\ 2/3 & -3/7 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 0 & 7 \end{bmatrix}.$$

**GS4.** Suppose that applying the Gram-Schmidt algorithm to $A = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}$ results in $A = QR$, where $Q = \begin{bmatrix} -1/2 & 1/\sqrt{2} \\ 1/2 & 0 \\ 1/2 & 1/\sqrt{2} \\ 1/2 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 4 & -2 \\ 0 & \sqrt{2} \end{bmatrix}$. (The matrix $Q$ satisfies $Q^T Q = I$.) Determine $x*$ that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^2$, where $b = \begin{bmatrix} -4 \\ 4 \\ 2 \\ 2 \end{bmatrix}$.

With $b = \begin{bmatrix} -4 \\ 4 \\ 2 \\ 2 \end{bmatrix}, Q^T b = \begin{bmatrix} 6 \\ -\sqrt{2} \end{bmatrix}$ and by solving $Rx^* = \begin{bmatrix} 2 \\ -\sqrt{2} \end{bmatrix}$, we get $x^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.**

$$r = Ax^* - b = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -4 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**GS5.** Suppose that applying the Gram-Schmidt algorithm to $A = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & 0 \end{bmatrix}$ results in

$A = QR$, where $Q = \begin{bmatrix} -2/3 & 1/\sqrt{5} \\ 1/3 & 2/\sqrt{5} \\ 2/3 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5}/3 \end{bmatrix}$. (The matrix $Q$ satisfies $Q^T Q = I$.)

Determine $x^*$ that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^2$, where $b = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$.

With $b = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$, $Q^T b = \begin{bmatrix} 3 \\ -2\sqrt{5} \end{bmatrix}$ and by solving $Rx^* = \begin{bmatrix} 3 \\ -2\sqrt{5} \end{bmatrix}$, we get $x^* = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

**Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.**

$$r = Ax^* - b = \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \end{bmatrix} - \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**True or False**

**TF1.** Answer true or false:

_True_ a. If $A$ is a $n \times n$ matrix, then $(A^2)^T = (A^T)^2$.

_False_ b. For $A$, an $n \times n$ matrix, t even if the equation $Ax = b$ has more than one solution (for some $Ax = b$), the transformation $x \mapsto Ax$ can be one-to-one.

_True_ **c.** If the columns of an $n \times n$ matrix are linearly independent then the columns must span $\mathbb{R}^n$.

_False_ **d.** If the matrices $A$ and $B$ are invertible, then the matrix $A + B$ is invertable.

_True_ **e.** If $A$ is a $n \times n$ matrix, then $(A^2)^T = (A^T)^2$.

_False_ **f.** For $A$, an $n \times n$ matrix, t even if the equation $Ax = b$ has more than one solution (for some $Ax = b$), the transformation $x \mapsto Ax$ can be one-to-one.

_True_ **g.** If the columns of an $n \times n$ matrix are linearly independent then the columns must span $\mathbb{R}^n$.

_False_ **h.** If the matrices $A$ and $B$ are invertible, then the matrix $A + B$ is invertable.

_False_ **i.** If the problem $Ax = b$ has any solution $x$, then the null space of must be only $\{0\}$.

_True_ **j.** If the problem $Ax = b$ has any solution $x$, then $b$ must be in the column space of $A$.

_False_ **k.** For any $m \times n$ matrix $A$, the null space of $A$ is a subspace of $\mathbb{R}^m$.

_True_ **l.** For any square matrix $A$, if the rows are linearly independent so are the columns.

**Vector Spaces**

**VS1.** Identify which of the following satisfy the definition for vector spaces. For each case mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

| | Closed under addition | Closed under scalar multiplication |
|---|---|---|
| **a.** The set of four by four matrices $A$ such that $A_{1,1} = 0$. | ___Yes___ | ___Yes___ |
| **b.** The set of polynomials of degree at most ten | ___Yes___ | ___Yes___ |
| **c.** The set of ordered triples of real numbers $(a,b,c)$ so that $\|a-1\| \le 1, \|b-1\| \le 1$, and $\|c-1\| \le 1$. | ___No____ | ___No____ |

The element $(2,2,2)$ is in the set but $(4,4,4) = (2,2,2) + (2,2,2)$ is not.
The element $(2,2,2)$ is in the set but $(4,4,4) = 2(2,2,2)$ is not.

| | | |
|---|---|---|
| **d.** The set of ordered pairs pairs of real numbers $(a,b)$ so that $ab \le 0$. | ___No____ | ___Yes___ |

The elements $(1,0)$ and $(0,1)$ are in the set but $(1,1) = (1,0) + (0,1)$ is not.

| | | |
|---|---|---|
| **e.** The set of infinite arrays $[x_1, x_2, x_3, \ldots]$ all of whose elements are non-negative | ___Yes___ | ___No____ |

The elements $[1,0,0,\ldots]$ is in the set but $[-1,0,0,\ldots] = -1[1,0,0,\ldots]$ is not.

| | | |
|---|---|---|
| **f.** The set of three by three upper triangular matrices. | ___Yes____ | ____Yes__ |
| **g.** The set of functions $f : [0,1] \to \mathbb{R}$ | ___Yes___ | ____Yes___ |
| **h.** The set of ordered pairs of real numbers $(a,b)$ so that $a \ge b$. | ___Yes___ | ____No___ |

$(1,0)$ is in the set but not $-1 \cdot (1,0) = (-1,0)$.

**i.** The set of ordered triples of real numbers $(a,b,c)$ so that $b=1$. ___No____ ___No____

$(0,1,0)$ is in the set but not $(0,1,0)+(0,1,0)=(0,2,0)$.

$(0,1,0)$ is in the set but not $-1\cdot(0,1,0)=(0,-1,0)$.

**j.** The set of two by two invertible matrices. ___No____ ___No___

$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ are in the set but not $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is in the set but not $0\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

**Null Spaces, Inverses, etc.**

**N1. a.** Find the null space of the matrix $A=\begin{bmatrix}4 & 2 & -3\\ 2 & 1 & 5\\ 2 & 1 & -2\end{bmatrix}$.

Since $\begin{bmatrix}4 & 2 & -3\\ 2 & 1 & 5\\ 2 & 1 & -2\end{bmatrix}\to\begin{bmatrix}4 & 2 & -3\\ 0 & 0 & 13/2\\ 0 & 0 & -1/2\end{bmatrix}$, a vector $z$ in the null space satisfies

$(-1/2)z_3=(13/2)z_3=0,$ and $z_1=\dfrac{0-2z_2-(-3)z_3}{4}=-z_2/2.$ With $z_2=1, z_1=-1/2$, so the null space is the set of all multiples of $\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix}$.

**b.** For the matrix $A$ in part a. find a solution to $Ax=\begin{bmatrix}-3\\ 1\\ 6\end{bmatrix}$ or show no solution exists.

If we do the same elimination operations to the right hand side as to the matrix we have $\begin{bmatrix}-3\\ 1\\ 6\end{bmatrix}\to\begin{bmatrix}-3\\ 5/2\\ 15/2\end{bmatrix}$ thus we have both $\dfrac{13}{2}x_3=\dfrac{5}{2}$ and $\dfrac{-1}{2}x_3=\dfrac{15}{2}$. The first gives $x_3=\dfrac{5}{13}$ and the second gives $x_3=-15$ so no solution exists.

**N2.** Display a matrix $A$ so that $A\begin{bmatrix}2\\ 0\end{bmatrix}=\begin{bmatrix}8\\ -10\end{bmatrix}$ and $A\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}3\\ 4\end{bmatrix}$. (Hint: If the solution is not obvious, determine some equations for the elements of $A$ and solve them.)

We know $A\begin{bmatrix}2\\ 0\end{bmatrix}=\begin{bmatrix}8\\ -10\end{bmatrix}$, so $A\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}4\\ -5\end{bmatrix}$, and the first column of $A$ must be $\begin{bmatrix}4\\ -5\end{bmatrix}$.

Since $A\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}3\\ 4\end{bmatrix}$, the sum of the first and second columns is $\begin{bmatrix}3\\ 4\end{bmatrix}$, so the second column is $\begin{bmatrix}3\\ 4\end{bmatrix}-\begin{bmatrix}4\\ -5\end{bmatrix}=\begin{bmatrix}-1\\ 9\end{bmatrix}$. Therefore, $A=\begin{bmatrix}4 & -1\\ -5 & 9\end{bmatrix}$.

**N3 a.** Find the null space of the matrix $A = \begin{bmatrix} 2 & 8 & 4 \\ 1 & 4 & 4 \\ -1 & -4 & -3 \end{bmatrix}$.

By elimination, we have $A = \begin{bmatrix} 2 & 8 & 4 \\ 1 & 4 & 4 \\ -1 & -4 & -3 \end{bmatrix} \to \begin{bmatrix} 2 & 8 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$. Since there is one zero on the diagonal, the null space must have dimension one. If $Az = 0$, we must have $(-1)z_3 = 0$ and $2z_3 = 0$ which imply $z_3 = 0$. But also $0 = 2z_1 + 8z_2 + 4z_3 = 2z_1 + 8z_2$, so with $z_2 = 1$ then $z_1 = -4$, and the nullspace is the set of all multiples of $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$.

**b.** For the matrix $A$ in part a. find a solution to $Ax = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ or show no solution exists.

If we do the same elimination operations to the right hand side as to the matrix we have $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ thus we have both $2x_3 = 2$ and $(-1)x_3 = -1$. Thus $x_3 = 1$ and the first equation gives $x_1 = \frac{-2 - 8x_2 - 4x_3}{2}$. with $x_2 = 0$ this gives $x_1 = \frac{-2-4}{2} = -3$ so a solution is $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.