Examination 1 Solutions
M340L-CS

1. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
2. Do not submit the scratch sheets. However, all of the work necessary to obtain the solution should be on these sheets.
3. Comment on all errors and omissions and enclose the **comments in boxes**

1. [20] Mark T(rue) or F(alse) for each of the following statements:

__F_a. A consistent system of linear equations could have no solution.

__T_b. The column space of A is the set of all vectors that can be written as Ax for some x .

__T_c. The solution set of the linear system whose augmented matrix is $\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{b} \end{bmatrix}$ is the same as the solution set of the equation $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 = \boldsymbol{b}$.

__F_d. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors, Span $\{\boldsymbol{u}, \boldsymbol{v}\}$ contains only the line through $\boldsymbol{u}$ and the origin, and the line through $\boldsymbol{v}$ and the origin.

__T_e. A vector $\boldsymbol{b}$ is a linear combination of the columns of a matrix $\boldsymbol{A}$ if and only if the equation $\boldsymbol{Ax} = \boldsymbol{b}$ has at least one solution.

__T_f. Any linear combination of vectors can always be written in the form $\boldsymbol{Ax}$ for a suitable matrix $\boldsymbol{A}$ and vector $\boldsymbol{x}$.

__F_g. The equation $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v}$ describes a line through $\boldsymbol{v}$ parallel to $\boldsymbol{p}$.

__F_h. The columns of a matrix $\boldsymbol{A}$ are linearly independent if the equation $\boldsymbol{Ax} = \boldsymbol{0}$ has the trivial solution.

__T_i. The columns of any 4×5 matrix are linearly dependent.

__T_j. If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \text{and} \boldsymbol{v}_5$ are in $\mathbb{R}^5$ and $\boldsymbol{v}_3 = \boldsymbol{0}$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ is linearly dependent.

2. Find the general solutions of the systems whose **augmented** matrix is:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 3 & -6 & -6 & 8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus x_4 is free, $x_3 = (3-1x_4)/3 = 1 - x_4/3,$ x_2 is free, and

$x_1 = (0-(-2)x_2 - (-1)x_3 - 3x_4)/1 = 1 + 2x_2 - 10x_4/3,$

3. Show that if $Ax = b$ and $Ay = b$, then for any scalar α, $A(\alpha(y-x)) = 0$.

If $Ax = b$ and $Ay = b$ then for any scalar α,

$$A\alpha(y-x) = \alpha A(y-x) = \alpha(Ay - Ax) = \alpha(b-b) = \alpha 0 = 0.$$

4. Suppose you are to solve *m* different linear systems of *n* equations in *n* unknowns. All of the equations have the same matrix, however, they just differ in right hand sides. Estimate how many ***additions*** are required.

The elimination step on the matrix requires $\frac{n^3}{3}$ additions and each of the m different linear systems requires n^2 additions for a total of $\frac{n^3}{3} + mn^2$ additions.

5. Use the Gaussian Elimination with Partial Pivoting and Solution algorithm to solve

$$-2x_1 - 2x_2 \quad + x_3 = 4$$
$$x_1 - 4x_2 \ + 6x_3 = 11$$
$$4x_1 - 8x_2 + 4x_3 = 4$$

Show what occupies storage in the A matrix and the ip array initially and after each major step of elimination.

A (initial):

-2	-2	1
1	-4	6
4	-8	4

ip: ?, ?, ?

After step 1:

4	-8	4
1/4	-2	5
-1/2	-6	3

ip: 3, ?, ?

After step 2:

4	-8	4
1/4	-6	3
-1/2	1/3	4

ip: 3, 3, ?

Final:

4	-8	4
1/4	-6	3
-1/2	1/3	4

ip: 3, 3, 3

For the elimination applied to $b = \begin{bmatrix} 4 \\ 11 \\ 4 \end{bmatrix}$ we get it changing to $\begin{bmatrix} 4 \\ 11 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix}$, and finally $\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$. In the bottom loop we $b_3 = 8$ and $x_3 = \frac{8}{4} = 2$. Then we get $b_2 = 6 - 3 \cdot 2 = 0$ so $x_2 = \frac{0}{-6} = 0$. Finally we get $b_1 = 4 - (-8) \cdot 0 - 4 \cdot 2 = -4$ so $x_1 = \frac{-4}{4} = -1$.

6. Fill in the five blanks in the code for Gaussian Elimination with Partial Pivoting and Solution separated from elimination:

for k = 1:n
  choose ip_k such that $|A_{ip_k,k}| = \max\{|A_{i,k}| : i \geq k\}$
    if $\underline{A_{ip_k,k} = 0}$
      warning ('Pivot in Gaussian Elimination is zero')
    end
    swap $A_{k,k},\ldots,A_{k,n}$ with $A_{ip_k,k},\ldots,A_{ip_k,n}$
    for i = $k+1$:n
      $A_{i,k}$ = $\underline{A_{i,k} / A_{k,k}}$
      for j = $\underline{k+1:n}$
        $A_{i,j} = \underline{A_{i,j} - A_{i,k} A_{k,j}}$
      end
    end
end
for k = 1:n
    swap b_k with b_{ip_k}
    for i = k+1:n
      $b_i = \underline{b_i - A_{i,k} b_k}$
    end
end
for i = n:-1:1
    for j = i+1:n
      $b_i = b_i - A_{i,j} x_j$
    end
    $x_i = b_i / A_{i,i}$
end