M340L-CS Exam 1

1. Either show that the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ or show that no such linear combination exists.

If we form the equations:

$$\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and find the row echelon form for the augmented matrix, we get:
$\begin{bmatrix} 2 & -4 & 1 \\ -1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -4 & 1 \\ 0 & 0 & 5/2 \end{bmatrix}$. The second equation is inconsistent, thus here are no solutions to the linear system and, thus, the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a not a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

2. Show that if $Ax = 0$ and $Ay = b$, then for any scalar $A(y + \alpha x) = b$.

If $Ax = 0$ and $Ay = b$, then for any scalar $A(y + \alpha x) = Ay + A\alpha x = Ay + \alpha Ax = b + \alpha 0 = b$.

3. Find the general solutions of the systems whose **augmented** matrix is:

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix}$$

If we find the row echelon form for the augmented matrix, we get:

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 and x_2 are free, $x_1 = (0 - (-2)x_2 - 4x_3)/3 = 2/3x_2 - 4/3x_3$.

4. Suppose the application of the Gaussian Elimination algorithm on a 50 by 50 matrix is timed at 500 μ seconds. How much time do you estimate would be required to apply the algorithm to a 2000 by 2000 matrix? (Ignore any issues with respect to the requirement for additional memory.)

> Our model for the computation time for the Gaussian Elimination algorithm is that it is essentially cubic in the size of the matrix (i.e. $n^3/3$ multiplications and $n^3/3$ additions). Thus if we inflate the size from 50 to 2000 (i.e. by a factor of 40) we expect the computation time to be inflated by 64,000 = 40^3. Thus, if 500 μ seconds were required for the 50 by 50 matrix, we expect 32 seconds for the 2000 by 2000 matrix.

5. Use the Gaussian Elimination with Partial Pivoting and Solution algorithm to solve

$$\begin{aligned} x_1 - 6x_2 + 4x_3 &= -9 \\ 2x_1 + 6x_2 + 5x_3 &= 15 \\ 3x_1 + 6x_2 &= 9 \end{aligned}$$

Show what occupies storage in the *A* matrix and the *ip* array initially and after each major step of elimination. (Note: the 3 by 1 array below is for the ip array.)

1	-6	4		?
2	6	5		?
3	6	0		?

3	6	0		3
2/3	2	5		?
1/3	-8	4		?

3	6	0		3
2/3	-8	4		3
1/3	-1/4	6		?

3	6	0		3
2/3	-8	4		3
1/3	-1/4	6		3

For the elimination applied to $b = \begin{bmatrix} -9 \\ 15 \\ 9 \end{bmatrix}$ we get it changing to $\begin{bmatrix} 9 \\ 15 \\ -9 \end{bmatrix}, \begin{bmatrix} 9 \\ 9 \\ -12 \end{bmatrix}, \begin{bmatrix} 9 \\ -12 \\ 9 \end{bmatrix}$, and finally $\begin{bmatrix} 9 \\ -12 \\ 6 \end{bmatrix}$. In the bottom loop we get $b_3 = 6$ and $x_3 = \frac{6}{6} = 1$. Then we get

$b_2 = -12 - 4 \cdot 1 = -16$ so $x_2 = \frac{-16}{-8} = 2.$ Finally we get $b_1 = 9 - 6 \cdot 2 - 0 \cdot 1 = -3$ so $x_1 = 3 = -1.$

6. Fill in the five blanks in the code for Gaussian Elimination with Partial Pivoting and Solution

for k = 1:n
  choose ip_k such that $|A_{ip_k,k}| = \max\{|A_{i,k}| : i \ge k\}$
  if $A_{ip_k,k}$ = 0
    warning ('Pivot in Gaussian Elimination is zero')
  end
  swap $A_{k,k},\ldots,A_{k,n}$ **with** $A_{ip_k,k},\ldots,A_{ip_k,n}$
  for i = k+1:n
    $A_{i,k} = \underline{A_{i,k} / A_{k,k}}$
    for j = $\underline{k+1:n}$
      $A_{i,j} = \underline{A_{i,j} - A_{i,k}A_{k,j}}$
    end
  end
end
for k = 1:n
  swap b_k **with** b_{ip_k}
  for i = k+1:n
    $b_i = \underline{b_i - A_{i,j}b_k}$
  end
end
for i = n:-1:1
  for j = i+1:n
    $b_i = b_i - A_{i,j}x_j$
  end
  $x_i = \underline{b_i / A_{i,i}}$
end

7. Answer T(rue) or F(alse):

__F_a. A consistent system of linear equations could have no solution.

__F_b. The column space of A is the set of all solutions of $Ax = b$.

__T_c. If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $Ax = b$ is consistent for each b in $\mathbb{R}^m$.

__F_g. If A and B are $m \times n$, then AB^T is defined but $A^T B$ is not defined.

__F_h. If $AC = 0$, then either $A = 0$ or $C = 0$.

__T_i. The column space of A is the set of all vectors that can be written as Ax for some x.

__T_j. If the problem $Ax = b$ has a solution x, then the problem $HAx = Hb$ has the same solution x, for any matrix H (for which HA and Hb are defined.)

__T_k. Let matrix A have columns $\begin{bmatrix} A_{\cdot,1} & \cdots & A_{\cdot,n} \end{bmatrix}$ and the problem $Ax = b$ has a solution x, then $b = x_1 A_{\cdot,1} + \ldots + x_n A_{\cdot,n}$.

__F_l. _ If the $n \times n$ matrix A has an inverse B then $AB = I$ but BA does not necessarily equal I.

__T_l. _ For some $n \times n$ matrices A and B, $AB \neq BA$.