Solutions to Examination 2

1. [20] a. Suppose u and v are n-vectors such that $v^T u = 1$. Using the definition of a projection P, (i.e., $P = P^2$), prove that $P = uv^T$ is a projection.

We have:

$$\begin{aligned} P^2 &= (uv^T)(uv^T) \\ &= uv^T uv^T \\ &= u(v^T u)v^T \\ &= uv^T \\ &= P. \end{aligned}$$

b. Consider the partitioned matrix $U = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where A, B, and C are square and A and C are invertible. Show that, in partitioned form, $U^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$. (**Hint:** recall that in partitioned form $I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$.)

$$\begin{aligned} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix} &= \begin{bmatrix} AA^{-1} + B0 & -AA^{-1}BC^{-1} + BC^{-1} \\ -0A^{-1} + C0 & -0A^{-1}BC^{-1} + CC^{-1} \end{bmatrix} \\ &= \begin{bmatrix} AA^{-1} & -AA^{-1}BC^{-1} + BC^{-1} \\ 0 & CC^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & -BC^{-1} + BC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= I. \end{aligned}$$

c. Using the definition of an inverse and the properties of the transpose of matrix products, prove for any invertible matrix A, that $(A^{-1})^T = (A^T)^{-1}$.

Since $A^{-1}A = I$, we have $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$, thus $(A^{-1})^T = (A^T)^{-1}$.

2. [15] Given $A = \begin{bmatrix} 6 & 1 \\ 0 & -4 \\ 8 & 8 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR$, where $Q^T Q = I$ and R is upper triangular.

$$R_{1,1} = \left\| \begin{matrix} 6 \\ 0 \\ 8 \end{matrix} \right\| = 10,\ Q_{.,1} = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}, R_{1,2} = Q_{.,1}^T A_{.,2} = 7,\ \bar{Q}_{.,2} = A_{.,2} - R_{1,2} Q_{.,1} = \begin{bmatrix} -16/5 \\ -20/5 \\ 12/5 \end{bmatrix},$$

$$R_{2,2} = \|\bar{Q}_{.,2}\| = 4\sqrt{2}, Q_{.,2} = \begin{bmatrix} -2\sqrt{2}/5 \\ -\sqrt{2}/2 \\ 3\sqrt{2}/10 \end{bmatrix}. \text{Thus, } A = \begin{bmatrix} 3/5 & -2\sqrt{2}/5 \\ 0 & -\sqrt{2}/2 \\ 4/5 & 3\sqrt{2}/10 \end{bmatrix} \begin{bmatrix} 10 & 7 \\ 0 & 4\sqrt{2} \end{bmatrix}.$$

2. [10] Answer true or false:

false **a.** If A is a 3×2 matrix, then the transformation $x \mapsto Ax$ cannot be one-to-one.

_true _ **b.** A transformation T is linear if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for all x and y
in the domain of T and for all scalars α and β.

_false _ **c.** For matrices A and B for which the product AB is defined, the first column of AB is the first row of A times the matrix B.

_true _ **d.** If the matrix A is invertible, then the inverse of A^{-1} is A itself.

4. [10] Prove that if $A^T(Ax^* - b) = 0$, for some $x^* \in \mathbb{R}^n$, then for all $y \in \mathbb{R}^n$ $\|A(x^* + y) - b\|^2 \geq \|Ax^* - b\|^2$. (Hint: Consider the Pythagorean Theorem and the quantity $y^T A^T (Ax^* - b)$.)

For any $y \in \mathbb{R}^n, (Ay)^T (Ax^* - b) = y^T A^T (Ax^* - b) = y^T 0 = 0$, so

$\|A(x^* + y) - b\|^2 = \|(Ax^* - b) + Ay\|^2 = \|Ax^* - b\|^2 + \|Ay\|^2$, and since $\|Ay\|^2 \geq 0$,
$\|A(x^* + y) - b\|^2 \geq \|Ax^* - b\|^2$.

5. [20] Identify which of the following satisfy the definition for vector spaces. For each case mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

	Closed under addition	Closed under scalar multiplication
a. The set of four by four matrices A such that $A_{1,1} = 0$.	___Yes___	___Yes___
b. The set of polynomials of degree at most ten	___Yes___	___Yes___
c. The set of ordered triples of real numbers (a,b,c) so that $\lvert a-1\rvert \le 1, \lvert b-1\rvert \le 1$, and $\lvert c-1\rvert \le 1$.	___No____	___No____

The element $(2,2,2)$ is in the set but $(4,4,4) = (2,2,2) + (2,2,2)$ is not.
The element $(2,2,2)$ is in the set but $(4,4,4) = 2(2,2,2)$ is not.

	Closed under addition	Closed under scalar multiplication
d. The set of ordered pairs pairs of real numbers (a,b) so that $ab \le 0$.	___No____	___Yes___

The elements $(1,0)$ and $(0,1)$ are in the set but $(1,1) = (1,0) + (0,1)$ is not.

	Closed under addition	Closed under scalar multiplication
e. The set of infinite arrays $[x_1, x_2, x_3, \ldots]$ all of whose elements are non-negative	___Yes___	___No____

The elements $[1,0,0,\ldots]$ is in the set but $[-1,0,0,\ldots] = -1[1,0,0,\ldots]$ is not.

6. [15] Find the nullspace of $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 4 \end{bmatrix}$ (i.e., find the general solution of $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$) or show that no nullspace exists.

To solve the system $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we transform $\begin{bmatrix} 3 & 1 & 2 & 0 \\ -3 & -3 & -1 & 0 \\ 6 & 2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and get that is x_3 free, $-2x_2 + x_3 = 0$, so $x_2 = \frac{1}{2}x_3 = 0$, and $3x_1 + x_2 + 2x_3 = 0$, so $x_1 = \frac{1}{3}(-x_2 - 2x_3) = \frac{1}{3}(-\frac{1}{2}x_3 - 2x_3) = -\frac{5}{6}x_3$. So, the nullspace is the set of vectors of the form $\begin{bmatrix} -\frac{5}{6}x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -\frac{5}{6} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ for some x_3.

7. [15] Suppose that applying the Gram-Schmidt algorithm to $A = \begin{bmatrix} -1 & 4 \\ 1 & 0 \\ 1 & -2 \\ 1 & -2 \end{bmatrix}$ results in $A = QR$,

where $Q = \begin{bmatrix} -1/2 & 1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} \\ 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & -4 \\ 0 & 2\sqrt{2} \end{bmatrix}$. (The matrix Q satisfies $Q^T Q = I$.) Determine

x^* that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^2$, where $b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \end{bmatrix}$.

With $b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \end{bmatrix}, Q^T b = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and by solving $Rx^* = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$, we get $x^* = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$.

Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.

$$r = Ax^* - b = \begin{bmatrix} -1 & 4 \\ 1 & 0 \\ 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -3/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \begin{bmatrix} -1 & 4 \\ 1 & 0 \\ 1 & -2 \\ 1 & -2 \end{bmatrix}^T \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$