1	10	
2	10	
3	15	
4	5	
5	10	
6	10	
7	20	
8	10	
9	20	
10	25	
11	10	
12	10	
13	15	
Total	170	

Name ______________________________

Final Examination

M340L-CS

1. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
2. Do not submit the scratch sheets. However, all of the work necessary to obtain the solution should be on these sheets.
3. Comment on all errors and omissions and enclose the **comments in boxes**

1. [10] Find a vector z that is perpendicular to $\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$ and has norm equal to two.

2. [10] Suppose that just prior to the second step of Gaussian elimination (with partial pivot selection) applied to a matrix A, we have in storage:

A

2	5	3	-1
1/2	-4	0	9
-1/3	2	8	5
0	-8	12	8

ip

4
?
?
?

What is found in A and ip immediately after the completion of the second step?

3. [15] Find the inverse of $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 3 \end{bmatrix}$ or show that no inverse exists.

4. [5] Find two linearly independent vectors in the null space of $\begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \\ -3 & -6 & 3 \end{bmatrix}$.

5. [10] Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR$, where $Q^T Q = I$ and R is upper triangular.

6. [10] Suppose that applying the Gram-Schmidt algorithm to $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -2 & -1 \\ 0 & 1 \end{bmatrix}$ results in $A = QR$,

where $Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 2/\sqrt{5} \\ -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{5} \end{bmatrix}$ and $R = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ 0 & \sqrt{5} \end{bmatrix}$. (The matrix Q satisfies $Q^T Q = I$.)

Determine x^* that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^2$, where $b = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$.

Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.

7. [20] Identify which of the following satisfy the definition for vector spaces. For each case either mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

	Closed under addition	Closed under scalar multiplication
a. The set of four by four matrices A such that $A_{4,1} = A_{1,4} = 0$.	______	______
b. The set of functions $f:[0,1]\to[0,\infty)$ (that is, the set of functions mapping the interval [0, 1] into non-negative values). .	______	______
c. The set of ordered triples of real numbers (a,b,c) so that $\lvert a-1\rvert\le 1, \lvert b-1\rvert\le 1$, and $\lvert c-1\rvert\le 1$.	______	______
d. The set of ordered pairs of real numbers (a,b) so that $ab\ge 0$.	______	______
e. The set of four by four matrices all of whose elements are non-negative.	______	______

8. [10] Answer true or false:

______ **a.** If the problem $HAx = Hb$ has a solution x, then the problem $Ax = b$ has the same solution x, for any matrix H (for which HA and Hb are defined.)

______ **b.** Let matrix A have rows $\begin{bmatrix} A_{1,\cdot} \\ \vdots \\ A_{n,\cdot} \end{bmatrix}$ and the problem $Ax = b$ has a solution x, then $b = x_1 A_{1,\cdot} + \ldots + x_n A_{n,\cdot}$.

______ **c.** If $p(\lambda) = \det(A - \lambda I)$ then the roots of $p(\lambda) = 0$ must contain all of the eigenvalues of A, and with the same multiplicity.

______ **d.** If $BAB^{-1} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ for an invertible matrix B then the eigenvalues of A must be $a, d,$ and f.

______ **e.** If A is a 3×2 matrix, then the transformation $x \mapsto Ax$ cannot be one-to-one.

NOTE: Scale all eigenvectors so the maximum element is 1.

9. [20] a. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors?

b. Let $B = \begin{bmatrix} 2 & 1 \\ 0 & 2+\varepsilon \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Your answers should be in terms of the perturbation parameter ε.)

c. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of A. Comment on the linear independence of the eigenvectors of B.

d. Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Choose linearly independent eigenvectors.)

e. Let $D = \begin{bmatrix} 2 & \varepsilon \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors?

f. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of C.

10. [25] Prove:

a. Using the definition of an inverse, prove for any invertible matrix A, that $(A^{-1})^T = (A^T)^{-1}$.

b. Prove that if λ is an eigenvalue of an invertible matrix A, then λ^2 is an eigenvalue of A^2.

c. For given n by n matrices A and B (where B is non-singular) if λ is an eigenvalue of A with associated eigenvector v then λ is an eigenvalue of BAB^{-1} with associated eigenvector Bv.

d. Prove that if A is not invertible then then 0 is an eigenvalue of A.

e. Prove that if $y^T x = 0$, for all x, then $y = 0$.

11. [10] Consider the parameterized matrix $A(\alpha) = \begin{bmatrix} 3 & \alpha \\ 7 & 2 \end{bmatrix}$.

a. For what real values of α does $A(\alpha)$ have two real and distinct eigenvalues?

b. For what real values of α does $A(\alpha)$ have one real eigenvalue of multiplicity two?

c. For what real values of α does $A(\alpha)$ have two eigenvalues that are not real?

12. [10] Find an eigenvalue and corresponding eigenvector of the n by n lower triangular matrix

$$A = \begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{bmatrix}$$

(Hint Problem 13 can easily be solved without finding the power of ANY matrix.)

13. [15] a. Find the eigenvalues of $E = \begin{bmatrix} 3 & 4 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

b. Find corresponding eigenvectors of E.

c. Assume if $Ex = \lambda x$ then for any integer $k, E^k x = \lambda^k x.$ Find the eigenvalues of $\bar{E} = E^3 - 4E + 2I - 2E^{-1}$.

d. Find corresponding eigenvectors of $\bar{E}$.