M340L-CS
Solutions

1. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
2. Do not submit the scratch sheets. However, all of the work necessary to obtain the solution should be on these sheets.
3. Comment on all errors and omissions and enclose the **comments in boxes**

1. [10] Find a vector z that is perpendicular to $\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$ and has norm equal to two.

We want $\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}^T z = -4z_1 + 2z_2 + 1z_3 - 2z_5 = 0$ and $\|z\| = 2$. One vector perpendicular to $\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. The norm of that is 1, so by multiplying by 2 we have a vector of norm 2.

Thus we get $z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$.

2. [10] Suppose that just prior to the second step of Gaussian elimination (with partial pivot selection) applied to a matrix *A*, we have in storage:

A

2	5	3	-1
1/2	-4	0	9
-1/3	2	8	5
0	-8	12	8

ip

4
?
?
?

What is found in A and ip immediately after the completion of the second step?

First, we determine the pivot in the third row so ip_2 is 3. We swap the second and third rows (from the second column) to get:

A

2	5	3	-1
1/2	-8	12	8
-1/3	2	8	5
0	-4	0	9

ip

4
4
?
?

Second, we compute the multipliers in the second column, and then update the elements in the lower right two by two sub-matrix.

A

2	5	3	-1
1/2	-8	12	8
-1/3	-1/4	11	7
0	1/2	-6	5

ip

4
4
?
?

3. [15] Find the inverse of $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 3 \end{bmatrix}$ or show that no inverse exists.

We need to solve the systems $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for the columns of the inverse matrix $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$. We transform $\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ -3 & -3 & -1 & 0 & 1 & 0 \\ 6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$. If we backsolve $\begin{bmatrix} 3 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, we get $\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} -7/6 \\ 1/2 \\ 2 \end{bmatrix}$. If we backsolve $\begin{bmatrix} 3 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, we get $\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/2 \\ 0 \end{bmatrix}$. Finally, if we backsolve $\begin{bmatrix} 3 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we get $\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 5/6 \\ -1/2 \\ -1 \end{bmatrix}$.

The inverse is then $\begin{bmatrix} -7/6 & 1/6 & 5/6 \\ 1/2 & -1/2 & -1/2 \\ 2 & 0 & -1 \end{bmatrix}$

4. [5] Find two linearly independent vectors in the null space of $\begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \\ -3 & -6 & 3 \end{bmatrix}$.

On elimination we get $\begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \\ -3 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$ are in the null space.

5. [10] Given $A=\begin{bmatrix}1 & -1\\ 2 & 0\\ 2 & 2\end{bmatrix}$, use the Gram Schmidt algorithm to express $A=QR$, where $Q^TQ=I$ and R is upper triangular.

$$R_{1,1}=\left\|\begin{bmatrix}1\\2\\2\end{bmatrix}\right\|=3,\ Q_{.,1}=\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}, R_{1,2}=Q_{.,1}^TA_{.,2}=1,\ \bar{Q}_{.,2}=A_{.,2}-R_{1,2}Q_{.,1}=\begin{bmatrix}-4/3\\-2/3\\4/3\end{bmatrix},$$

$$R_{2,2}=\left\|\bar{Q}_{.,2}\right\|=2, Q_{.,2}=\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix}.\text{Thus, } A=\begin{bmatrix}1/3 & -2/3\\ 2/3 & -1/3\\ 2/3 & 2/3\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}.$$

6. [10] Suppose that applying the Gram-Schmidt algorithm to $A=\begin{bmatrix}2 & 1\\ 0 & 2\\ -2 & -1\\ 0 & 1\end{bmatrix}$ results in $A=QR$,

where $\underset{\sim}{Q}=\begin{bmatrix}1/\sqrt{2} & 0\\ 0 & 2/\sqrt{5}\\ -1/\sqrt{2} & 0\\ 0 & 1/\sqrt{5}\end{bmatrix}$ and $R=\begin{bmatrix}2\sqrt{2} & \sqrt{2}\\ 0 & \sqrt{5}\end{bmatrix}$. (The matrix $\underset{\sim}{Q}$ satisfies $\underset{\sim}{Q}^T\underset{\sim}{Q}=I$.)

Determine x^* that minimizes $\|Ax-b\|$ over all $x\in\mathbb{R}^2$, where $b=\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix}$.

With $b=\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix}, Q^Tb=\begin{bmatrix}\sqrt{2}\\ -8/\sqrt{5}\end{bmatrix}$ and by solving $Rx^*=\begin{bmatrix}\sqrt{2}\\ -8/\sqrt{5}\end{bmatrix}$, we get $x^*=\begin{bmatrix}3/10\\ -8/5\end{bmatrix}$.

Hint: A smart student would now test $A^T r$, where $r = Ax^* - b$.

$$r = Ax^* - b = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3/10 \\ -8/5 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -6/5 \\ -2 \\ 12/5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -2 & -1 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} -2 \\ -6/5 \\ -2 \\ 12/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

7. [20] Identify which of the following satisfy the definition for vector spaces. For each case either mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

	Closed under addition	Closed under scalar multiplication
a. The set of four by four matrices A such that $A_{4,1} = A_{1,4} = 0$.	__Yes__	__Yes__

b. The set of functions $\{f : [0,1] \to [0,\infty)\}$ (that is, the set of functions mapping the interval [0, 1] into non-negative values). . — __Yes__ — __No__

The function $f(x) = 1$, for $x \in [0,1]$, is in the set $\{f : [0,1] \to [0,\infty)\}$ but $-f(x) = -1$, for $x \in [0,1]$ is not.

c. The set of ordered triples of real numbers (a,b,c) so that $|a-1| \le 1, |b-1| \le 1$, and $|c-1| \le 1$. — __No__ — __No__

The element $(2,2,2)$ is in the set but $(4,4,4) = (2,2,2) + (2,2,2)$ is not.
The element $(2,2,2)$ is in the set but $(4,4,4) = 2(2,2,2)$ is not.

d. The set of ordered pairs of real numbers (a,b) so that $ab \ge 0$. — __No__ — __Yes__

The elements $(1,0)$ and $(0,-1)$ are in the set but $(1,-1) = (1,0) + (0,-1)$ is not.

e. The set of four by four matrices all of whose elements are non-negative. __Yes__ __No__

The element $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in the set but $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (-1)\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not.

8. [10] Answer true or false:

False **a.** If the problem $HAx = Hb$ has a solution x, then the problem $Ax = b$ has the same solution x, for any matrix H (for which HA and Hb are defined.)

False **b.** Let matrix A have rows $\begin{bmatrix} A_{1,\cdot} \\ \vdots \\ A_{n,\cdot} \end{bmatrix}$ and the problem $Ax = b$ has a solution x, then $b = x_1 A_{1,\cdot} + \ldots + x_n A_{n,\cdot}$.

True **c.** If $p(\lambda) = \det(A - \lambda I)$ then the roots of $p(\lambda) = 0$ must contain all of the eigenvalues of A, and with the same multiplicity.

True **d.** If $BAB^{-1} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ for an invertible matrix B then the eigenvalues of A must be $a, d,$ and f.

False **e.** If A is a 3×2 matrix, then the transformation $x \mapsto Ax$ cannot be one-to-one.

NOTE: Scale all eigenvectors so the maximum element is 1.

9. [20] a. Let $A=\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$. What are its eigenvalues and eigenvectors?

The eigenvalues are 2 and 2. The null space of $A-2I=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\0\end{bmatrix}$ (and its multiples). $\begin{bmatrix}1\\0\end{bmatrix}$ is only one linearly independent eigenvector.

b. Let $B=\begin{bmatrix}2 & 1\\0 & 2+\varepsilon\end{bmatrix}$. What are its eigenvalues and eigenvectors? (Your answers should be in terms of the perturbation parameter ε.)

The eigenvalues are 2 and $2+\varepsilon$. The null space of $B-2I=\begin{bmatrix}0 & 1\\0 & \varepsilon\end{bmatrix}$ is the vector $\begin{bmatrix}1\\0\end{bmatrix}$ (and its multiples). The null space of $B-(2+\varepsilon)I=\begin{bmatrix}-\varepsilon & 1\\0 & 0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\\varepsilon\end{bmatrix}$ (and its multiples).

Thus the eigenvectors are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\\varepsilon\end{bmatrix}$.

c. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of *A*. Comment on the linear independence of the eigenvectors of *B*.

The perturbation has introduced a second eigenvector but it is nearly linearly dependent upon the first.

d. Let $C=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$. What are its eigenvalues and eigenvectors? (Choose linearly independent eigenvectors.)

The eigenvalues are 2 and 2. The null space of $C-2I=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ is all of $\mathbb{R}^2$. Two linearly independent eigenvectors are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

e. Let $D = \begin{bmatrix} 2 & \varepsilon \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors?

The eigenvalues are 2 and 2. The null space of $D - 2I = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and its multiples). The only eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

f. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of C.

The perturbation has left the eigenvalues unperturbed but has removed the second eigenvector.

10. [25] Prove:

a. Using the definition of an inverse, prove for any invertible matrix A, that $(A^{-1})^T = (A^T)^{-1}$.

We have $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$, thus $(A^{-1})^T = (A^T)^{-1}$.

b. Prove that if λ is an eigenvalue of an invertible matrix A, then λ^2 is an eigenvalue of A^2.

If λ is an eigenvalue of A, then there exists a non-zero vector x so that $Ax = \lambda x$. But then $A^2x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$. Thus, λ^2 is an eigenvalue of A^2.

c. For given n by n matrices A and B (where B is non-singular) if λ is an eigenvalue of A with associated eigenvector v then λ is an eigenvalue of BAB^{-1} with associated eigenvector Bv.

$(BAB^{-1})Bv = BA(B^{-1}B)v = BAIv = BAv = B\lambda v = \lambda Bv$, so λ is an eigenvalue of BAB^{-1} with associated eigenvector Bv.

d. Prove that if A is not invertible then then 0 is an eigenvalue of A.

Since A is not invertible, there exists an non-zero x so that $Ax = 0$, but then $Ax = 0x$. Thus 0 is an eigenvalue of A.

e. Prove that if $y^T x = 0$, for all x, then $y = 0$.

Since $y^T x = 0$, for all x, then in particular for $x = y, \|y\|^2 = y^T y = 0$, and if $\|y\| = 0, y = 0$.

11. [10] Consider the parameterized matrix $A(\alpha)=\begin{bmatrix} 3 & \alpha \\ 7 & 2 \end{bmatrix}$.

a. For what real values of α does $A(\alpha)$ have two real and distinct eigenvalues?

We have as the characteristic polynomial $\det(A(\alpha)-\lambda I)=\det(\begin{bmatrix} 3-\lambda & \alpha \\ 7 & 2-\lambda \end{bmatrix})=(3-\lambda)(2-\lambda)-7\alpha=\lambda^2-5\lambda+6-7\alpha$. The roots of this are $\lambda=\dfrac{5\pm\sqrt{25-4\cdot(6-7\alpha)}}{2}=\dfrac{5\pm\sqrt{1+28\alpha}}{2}$. If $\alpha>\dfrac{-1}{28}$ the discriminate is positive and the two roots are real and distinct.

b. For what real values of α does $A(\alpha)$ have one real eigenvalue of multiplicity two?

If $\alpha=\dfrac{-1}{28}$ the discriminate is zero and the root are real and of multiplicity two.

c. For what real values of α does $A(\alpha)$ have two eigenvalues that are not real?

If $\alpha<\dfrac{-1}{28}$ the discriminate is negative and the two roots are not real.

12. [10] Find an eigenvalue and corresponding eigenvector of the *n* by *n* lower triangular matrix

$$A=\begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{bmatrix}$$

We have $Ae_n=\begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ a_{n,n} \end{bmatrix}=a_{n,n}\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}=a_{n,n}e_n$, so $a_{n,n}$ is an eigenvalue with corresponding eigenvector e_n.

(Hint Problem 13 can easily be solved without finding the power of ANY matrix.)

13. [15] a. Find the eigenvalues of $E = \begin{bmatrix} 3 & 4 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

This is upper triangular. The eigenvalues are 3, -1, and 2.

b. Find corresponding eigenvectors of E.

The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

c. Assume if $Ex = \lambda x$ then for any integer $k, E^k x = \lambda^k x.$ Find the eigenvalues of $\bar{E} = E^3 - 4E + 2I - 2E^{-1}$.

The eigenvalues of $\bar{E} = E^3 - 4E + 2I - 2E^{-1}$ are

$3^3 - 4\cdot 3 + 2 - 2(1/3) = 16\frac{1}{3}, (-1)^3 - 4\cdot(-1) + 2 - 2(1/-1) = 7,$ and

$2^3 - 4\cdot 2 + 2 - 2(1/2) = 1.$

d. Find corresponding eigenvectors of $\bar{E}$.

The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.