Simple Gaussian Elimination Example

We seek to solve the system:

$$\begin{aligned} 3x_1 + 2x_2 - 2x_3 &= -5 \\ -2x_1 - x_2 + 3x_3 &= 6 \\ 6x_1 \quad\;\; - 12x_3 &= -18 \end{aligned}$$

The augmented matrix version of this is

$$\begin{bmatrix} 3 & 2 & -2 & -5 \\ -2 & -1 & 3 & 6 \\ 6 & 0 & -12 & -18 \end{bmatrix}$$

After multiplying the first row by $\frac{-2}{3}$ and subtracting that from the second row, we obtain:

$$\begin{bmatrix} 3 & 2 & -2 & -5 \\ 0 & \frac{1}{3} & \frac{5}{3} & \frac{8}{3} \\ 6 & 0 & -12 & -18 \end{bmatrix}$$

And we scale the second row by 3 to get:

$$\begin{bmatrix} 3 & 2 & -2 & -5 \\ 0 & 1 & 5 & 8 \\ 6 & 0 & -12 & -18 \end{bmatrix}.$$

After multiplying the first row by $\frac{6}{3} = 2$ and subtracting that from the third row, we obtain:

$$\begin{bmatrix} 3 & 2 & -2 & -5 \\ 0 & 1 & 5 & 8 \\ 0 & -4 & -8 & -8 \end{bmatrix}$$

The first variable has now been eliminated from the last two equations. We will now eliminate the second variable from the third equation. After multiplying the second row by $\frac{-4}{1} = -4$ and subtracting that from the third row, we obtain:

$$\begin{bmatrix} 3 & 2 & -2 & -5 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 12 & 24 \end{bmatrix}.$$

That completes the forward elimination. The matrix now has the upper triangular (plus the extra column) structure we seek.

Turning now to back substitution, the third equation is $12x_3 = 24$, so $x_3 = \frac{24}{12} = 2$. The second equation is $x_2 + 5x_3 = 8$, so $x_2 = \frac{8-5x_3}{1} = \frac{8-5\cdot 2}{1} = \frac{-2}{1} = -2$. Lastly, the first equation is $3x_1 + 2x_2 - 2x_3 = -5$, so $x_1 = \frac{-5-2x_2-(-2)x_3}{3} = \frac{-5-2\cdot(-2)-(-2)\cdot 2}{3} = \frac{3}{3} = 1$. That, completes the back substitution.