Gram-Schmidt Orthonormalization of a Matrix and Solution of the Least Squares Problem

We assume we are given an $m \times n$ matrix A, where $n \le m$, whose columns are $a_1, \ldots, a_n$. We seek to express these columns as linear combinations of orthonormal columns $q_1, \ldots, q_n$. Letting Q be the matrix with these columns, this results in $A = QR$, where

$q_1, \ldots, q_n$, the columns of Q, are orthonormal (so $Q^T Q = I$),
R is an $n \times n$ upper triangular matrix.

1. The algorithm uses these two procedures:

(norm, u) = *normalize* (u): inputs vector u, outputs *norm* = $\|u\|$, and overwrites the original u with $\frac{1}{norm} u$.

(c, v) = *projectn* (u, v): inputs vectors u and v (where $\|u\| = 1$), computes $c = u^T v$, and overwrite the original v with $v - cu$

Q= A
$(r_{1,1}, q_1) = normalize(q_1)$
for $j = 2:n$
  for $i = 1:j-1$
    $(r_{i,j}, q_j) = projectn(q_i, q_j)$
  $(r_{j,j}, q_j) = normalize(q_j)$

2. To solve the least squares problem:

Determine x^* that minimizes $\|Ax - b\|$ over all $x \in \mathbb{R}^n$.

a. Calculation of $c = Q^T b$

for $i = 1:n$
  $(c_i, b) = projectn(q_i, b)$

b. Solve upper triangular system for the Least Squares solution x^*

solve $Rx^* = c$