

Gram-Schmidt Orthogonalization and a least squares problem:

a. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, Use the Modified Gram-Schmidt algorithm to express $A = QR$,

where the 3×2 matrix Q has orthonormal columns and the 2×2 matrix R is upper-triangular.

b. Compute $c = Q^T b$ as specified by the algorithm.

c. Solve the least squares problem $\min_x \|Ax - b\|$ by solving $Rx^* = c$.

Numerical Solution

a. With $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$, the Modified Gram-Schmidt algorithm computes

$$A = \begin{bmatrix} 0.8165 & 0.2182 \\ 0.4082 & 0.4364 \\ -0.4082 & 0.8729 \end{bmatrix} \begin{bmatrix} 2.4495 & -1.6330 \\ 0 & 1.5275 \end{bmatrix}.$$

b. With $b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $c = Q^T b = \begin{bmatrix} 1.2247 \\ 1.3093 \end{bmatrix}$

c. By solving $Rx^* = c$, we obtain the least squares solution $x^* = \begin{bmatrix} 1.0714 \\ 0.8571 \end{bmatrix}$.

Algebraic Solution

a. With $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$, the Modified Gram-Schmidt algorithm computes

$$A = \begin{bmatrix} \sqrt{6}/3 & \sqrt{21}/21 \\ \sqrt{6}/6 & 2\sqrt{21}/21 \\ -\sqrt{6}/6 & 4\sqrt{21}/21 \end{bmatrix} \begin{bmatrix} \sqrt{6} & -2\sqrt{6}/3 \\ 0 & \sqrt{21}/3 \end{bmatrix}.$$

b. With $b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $c = Q^T b = \begin{bmatrix} \sqrt{6}/3 \\ 2\sqrt{21}/7 \end{bmatrix}$

c. By solving $Rx^* = c$, we obtain the least squares solution $x^* = \begin{bmatrix} 15/14 \\ 6/7 \end{bmatrix}$.