

M 340L – CS
Homework Set 10 Solutions

1. Calculate the determinants of

$$\text{a. } \begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix}, \det \begin{bmatrix} 3 & 6 \\ -1 & 4 \end{bmatrix} = 3 \cdot 4 - 6 \cdot (-1) = 18.$$

$$\begin{aligned} \text{b. } & \begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & -1 \end{bmatrix}, \det \begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & -1 \end{bmatrix} \\ &= 2 \cdot 0 \cdot (-1) - 2 \cdot 3 \cdot 3 - (-2) \cdot 2 \cdot (-1) + (-2) \cdot 3 \cdot 4 + 4 \cdot 2 \cdot 3 - 4 \cdot 0 \cdot 4 = -22. \end{aligned}$$

2. The expansion of a 3×3 determinant can be remembered by the following device. Add a copy of the first two columns to the right of the matrix, and compute the determinant adding the products along the northwest-to-southeast diagonals and subtracting the products along the northeast-to-southwest diagonals:

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix} - \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

Use this method to compute the determinants:

$$\begin{aligned} \text{a. } & \begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix}, \det \begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \\ &= 0 \cdot (-3) \cdot 1 + 5 \cdot 0 \cdot 2 + 1 \cdot 4 \cdot 4 - 1 \cdot (-3) \cdot 2 - 0 \cdot 0 \cdot 4 - 5 \cdot 4 \cdot 1 = 2. \end{aligned}$$

$$\begin{aligned} \text{b. } & \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}, \det \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \\ &= 1 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 3 + 5 \cdot 2 \cdot 4 - 5 \cdot 1 \cdot 3 - 1 \cdot 1 \cdot 4 - 3 \cdot 2 \cdot 2 = 20. \end{aligned}$$

3. Prove that for an invertible matrix A , $\det(A^{-1}) = 1/\det(A)$. (Hint: Remember $AA^{-1} = I$.)

$$1 = \det(I) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1}), \text{ so } \det(A^{-1}) = 1/\det(A).$$

4. Answer true or false to the following. If false offer a counterexample.

a. If the columns of A are linearly dependent, then $\det(A) = 0$.

True. If the columns of A are linearly dependent, then A is singular and $\det(A) = 0$.

b. $\det(A+B) = \det(A)\det(B)$.

False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then
 $\det(A+B) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0 = 0 \cdot 0 = \det(A)\det(B)$.

c. The determinant of A is the product of the diagonal entries in A .

False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\det(A) = -1 \neq 0 \cdot 0$.

d. If $\det(A)$ is zero, then two rows or two columns are the same, or a row or a column is zero.

False. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$, then $\det(A) = 0$ but no two rows nor two columns are the same, nor is a row or a column zero.

5. Answer true or false to the following. If false offer a counterexample.

a. If $Ax = \lambda x$ for some scalar λ , then x is an eigenvector of A .

False. Let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x = \begin{bmatrix} 0 \end{bmatrix}$ then $Ax = 0 = 0 \cdot 0 = 0x$ but $x = \begin{bmatrix} 0 \end{bmatrix}$ is not an eigenvector of A .

b. If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A v_1 = 1 v_1$ and $A v_2 = 1 v_2$, so both v_1 and v_2 are linearly independent eigenvectors with the common eigenvalue 1.

c. The eigenvalues of a matrix are on its main diagonal.

False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $Av_1 = 1v_1$ and $Av_2 = -1v_2$, so the eigenvalues are 1 and -1, neither of which is on the diagonal.

6. For each of these matrices,

- find the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- factor it to get the eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_n$.
- for $i = 1, \dots, n$: find x^i the eigenvector corresponding λ_i . (that is, find a vector x^i in the nullspace of $A - \lambda_i I$).
- Scale all eigenvectors so the largest component is + 1.

a. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

The characteristic polynomial is $(1 - \lambda)(2 - \lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$ so the eigenvalues are 5 and -2. The null space of $A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}$ is the vector $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ (and its multiples). The null space of $A - (-2)I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. $A = \begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix}.$

The characteristic polynomial is

$$(8-\lambda)(9-\lambda)(3-\lambda) + (-12)(-3)(-5) + 0 - 0 - (8-\lambda)(-3)3 - (3-\lambda)(-3)(-12) \\ = -\lambda^3 + 20\lambda^2 - \lambda 96 = -(\lambda-8)(\lambda-12)(\lambda-0)$$

so the eigenvalues are 8, 12, and 0. The null space of $A-8I = \begin{bmatrix} 0 & -12 & 0 \\ -3 & 1 & -3 \\ -5 & 3 & -5 \end{bmatrix}$ is the

vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (and its multiples) since

$$\begin{bmatrix} 0 & -12 & 0 \\ -3 & 1 & -3 \\ -5 & 3 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 3 & -5 \\ -3 & 1 & -3 \\ 0 & -12 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 3 & -5 \\ 0 & -4/5 & 0 \\ 0 & -12 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 3 & -5 \\ 0 & -4/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ The null}$$

space of $A-12I = \begin{bmatrix} -4 & -12 & 0 \\ -3 & -3 & -3 \\ -5 & 3 & -9 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}$ (and its multiples) since

$$\begin{bmatrix} -4 & -12 & 0 \\ -3 & -3 & -3 \\ -5 & 3 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -12 & 0 \\ 0 & 6 & -3 \\ 0 & 18 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -12 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}. \text{ The null space of}$$

$A-0I = \begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}$ (and its multiples) since

$$\begin{bmatrix} 8 & -12 & 0 \\ -3 & 9 & -3 \\ -5 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -12 & 0 \\ 0 & 9/2 & -3 \\ 0 & -9/2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -12 & 0 \\ 0 & 9/2 & -3 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Thus the eigenvectors are } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2/3 \\ 1 \end{bmatrix}.$$

7. Two eigenvectors of an upper triangular matrix:

Let $U = \begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix}$ be n by n and upper triangular. Assume $a \neq c$.

a. Show that the eigenvector corresponding to the eigenvalue a is e_1 (i.e the first column of the n by n identity matrix). (Use this below.)

$$Ue_1 = \begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix} e_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = ae_1.$$

b. Show that the eigenvector corresponding to the eigenvalue c is $\begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}$. (Use this or a

scaled version $\begin{bmatrix} 1 \\ (c-a)/b \\ 0 \\ \vdots \end{bmatrix}$. below.)

$$U \begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} a & b & \cdots \\ 0 & c & \cdots \\ 0 & 0 & \ddots \end{bmatrix} \begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} ab/(c-a)+b \\ c \\ 0 \end{bmatrix} = c \begin{bmatrix} b/(c-a) \\ 1 \\ 0 \\ \vdots \end{bmatrix}.$$