M 340L – CS
Homework Set 11 Solutions

Note: Scale all eigenvectors so the largest component is + 1.

1. How do perturbations affect eigenvalues and eigenvectors?

a. Let $A=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}$. What are its eigenvalues and eigenvectors? (See the note above regarding the scaling of eigenvectors and make sure you do it throughout the homework.)

The eigenvalues are 2 and 2. The null space of $A-2I=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\ 0\end{bmatrix}$ (and its multiples). $\begin{bmatrix}1\\ 0\end{bmatrix}$ is only one linearly independent eigenvector.

b. Let $B=\begin{bmatrix}2 & 1\\ 0 & 2+\varepsilon\end{bmatrix}$. What are its eigenvalues and eigenvectors? (Your answers should be in terms of the perturbation parameter ε.)

The eigenvalues are 2 and $2+\varepsilon$. The null space of $B-2I=\begin{bmatrix}0 & 1\\ 0 & \varepsilon\end{bmatrix}$ is the vector $\begin{bmatrix}1\\ 0\end{bmatrix}$ (and its multiples). The null space of $B-(2+\varepsilon)I=\begin{bmatrix}-\varepsilon & 1\\ 0 & 0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\ \varepsilon\end{bmatrix}$ (and its multiples).

Thus the eigenvectors are $\begin{bmatrix}1\\ 0\end{bmatrix}$ and $\begin{bmatrix}1\\ \varepsilon\end{bmatrix}$.

c. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of A. Comment on the linear independence of the eigenvectors of B.

The perturbation has introduced a second eigenvector but it is nearly linearly dependent upon the first.

d. Let $C=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$. What are its eigenvalues and eigenvectors? (Choose linearly independent eigenvectors.)

The eigenvalues are 2 and 2. The null space of $C-2I=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is all of $\mathbb{R}^2$. Two linearly independent eigenvectors are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

e. Let $D=\begin{bmatrix}2&\varepsilon\\0&2\end{bmatrix}$. What are its eigenvalues and eigenvectors?

The eigenvalues are 2 and 2. The null space of $D-2I=\begin{bmatrix}0&\varepsilon\\0&0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\0\end{bmatrix}$ (and its multiples). The only eigenvector is $\begin{bmatrix}1\\0\end{bmatrix}$.

f. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of C.

The perturbation has left the eigenvalues unperturbed but has removed the second eigenvector.

2. Using the diagonal form to compute high powers:

Let $A=\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$. *Feel free to express answers in parts* **c**, **d**, *and* **e** *using expressions involving powers.*

a. What are its eigenvalues and eigenvectors?

The characteristic polynomial is $(1-\lambda)(1-\lambda)-4=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)$ so the eigenvalues are 3 and -1. The null space of $A-3I=\begin{bmatrix}-2&-2\\-2&-2\end{bmatrix}$ is the vector $\begin{bmatrix}1\\-1\end{bmatrix}$ (and its multiples). The null space of $A-(-1)I=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$ is the vector $\begin{bmatrix}1\\1\end{bmatrix}$ (and its multiples).

Thus the eigenvectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$.

b. Using part **a.**, form D, a diagonal matrix of eigenvalues, form V whose columns are the associated eigenvectors, then compute V^{-1}, and finally VDV^{-1}. Compare VDV^{-1} to A.

Since $\begin{bmatrix}1&1\\-1&1\end{bmatrix}^{-1}=\begin{bmatrix}1/2&-1/2\\1/2&1/2\end{bmatrix}$, $A=\begin{bmatrix}1&-2\\-2&1\end{bmatrix}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}1/2&-1/2\\1/2&1/2\end{bmatrix}$.

c. Using part **b.**, what is $A^{100}y$, for $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$? (Do **not** compute A^{100} - yet. Use associativity in a clever way.)

$$A^{100}y = VD^{100}V^{-1}y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3^{100}/2 \\ 3/2 \end{bmatrix}$$
$$= \begin{bmatrix} (3-3^{100})/2 \\ (3^{100}+3)/2 \end{bmatrix}$$

d. Express your answer in part **c** as $A^{100}y = \gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is + 1. Compare $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to the eigenvector corresponding to λ_1.

$$A^{100}y = \begin{bmatrix} (3-3^{100})/2 \\ (3^{100}+3)/2 \end{bmatrix} = \frac{(3^{100}+3)}{2} \begin{bmatrix} \frac{-3^{100}+3}{3^{100}+3} \\ 1 \end{bmatrix}$$. The vector $\begin{bmatrix} \frac{-3^{100}+3}{3^{100}+3} \\ 1 \end{bmatrix}$ is very close (within 10^{-46}) to the negative of the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

e. Using part **b.**, what is A^{100}?

$$A^{100} = VD^{100}V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3^{100}/2 & -3^{100}/2 \\ 1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} (3^{100}+1)/2 & (-3^{100}+1)/2 \\ (-3^{100}+1)/2 & (3^{100}+1)/2 \end{bmatrix}.$$

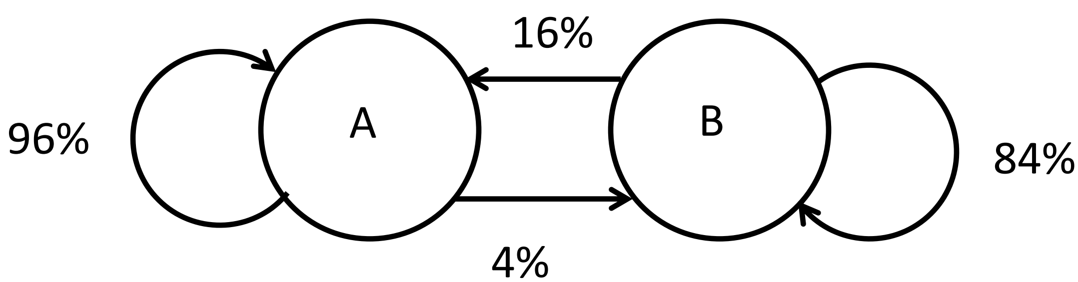
3. All zero eigenvalues:

Find a simple non-zero matrix having all zero eigenvalues.

The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a non-zero matrix having all zero eigenvalues

4. A Markov process:

A city has two restaurants: A and B. 96% of the time, a person leaving A will return to A the next time she goes to either A or B. Thus, 4% of the time, she will switch to B the next time. 84% of the time, a person leaving B will return to B the next time she goes to either A or B. Thus, 16% of the time, she will switch to A the next time. This diagram summarizes the situation:



Let $A = \begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix}$ (i.e. the matrix of transitions). use $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

a. What are its eigenvalues and eigenvectors?

The characte3ristic polynomial is $(24/25-\lambda)(21/25-\lambda)-4/25^2 = \lambda^2 - 9/5\lambda + 4/5 = (\lambda-1)(\lambda-4/5)$ so the eigenvalues are 1 and 4/5. The null space of $A - I = \begin{bmatrix} -1/25 & 4/25 \\ 1/25 & -4/25 \end{bmatrix}$. is the vector $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$(and its multiples). The null space of $A - 4/5I = \begin{bmatrix} 4/25 & 4/25 \\ 1/25 & 1/25 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$(and its multiples). Thus the eigenvectors are $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. Using part a. express $A = VDV^{-1}$ (where the columns of V are the eigenvectors and D is a diagonal matrix containing the associated eigenvalues.)

Since $\begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}$,

$A = \begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}$.

c. Using the fact that $A^k = VD^kV^{-1}$, what is $A^{100}y$, for $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$? (See comments in 1c.)

$$A^{100}y = VD^{100}V^{-1}y = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix} \begin{bmatrix} 4/5 \\ -3/10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 4/5 \\ -3 \cdot (4/5)^{100}/10 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 - 3 \cdot (4/5)^{100}/10 \\ 1/5 + 3 \cdot (4/5)^{100}/10 \end{bmatrix}$$

d. Express your answer in part **c** as $A^{100}y = \gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is + 1. Compare $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to the eigenvector corresponding to λ_1.

$$A^{100}y == \begin{bmatrix} 4/5 - 3 \cdot (4/5)^{100}/10 \\ 1/5 + 3 \cdot (4/5)^{100}/10 \end{bmatrix} = (4/5 - 3 \cdot (4/5)^{100}/10) \begin{bmatrix} 1 \\ \dfrac{1 + 15 \cdot (4/5)^{100}/10}{4 - 15 \cdot (4/5)^{100}/10} \end{bmatrix}.$$

The vector $\begin{bmatrix} 1 \\ \dfrac{1 + 15 \cdot (4/5)^{100}/10}{4 - 15 \cdot (4/5)^{100}/10} \end{bmatrix}$ is very close (within 10^{-10}) to the eigenvector $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$.

e. Using part **b**., what is A^{100}?

$$A^{100} = VD^{100}V^{-1} = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ (4/5)^{100}/5 & -(4/5)^{101} \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 + (4/5)^{100}/5 & 4/5 - (4/5)^{101} \\ 1/5 - (4/5)^{100}/5 & 1/5 + (4/5)^{101} \end{bmatrix}.$$