1. Mark T( rue) or F( alse) for each of the following statements:

_T_a. The column space of \( A \) is the set of all vectors that can be written as \( Ax \) for some \( x \).

_F_b. Elementary row operations on an augmented matrix can change the solution set of the associated linear system.

_T_c. If \( b \) is in the set spanned by the columns of \( A \) then the equation \( Ax = b \) is consistent.

_F_d. For an \( n \times n \) system of linear equations, the Gaussian Elimination Algorithm with Partial Pivoting and Elimination Separated from Solving uses approximately \( n^3 / 3 \) floating point multiplications and \( 2n^3 / 3 \) floating point additions/subtractions.

_T_e. the Gaussian Elimination Algorithm with Partial Pivoting has multipliers no larger than one in absolute value.

_T_f. A homogeneous equation is always consistent.

_F_g. The homogeneous equation \( Ax = 0 \) has the trivial solution if and only if the equation has at least one free variable.

_F_h. If \( x \) is a nontrivial solution of \( Ax = 0 \), then every entry in \( x \) is nonzero.

_T_i. The effect of adding \( p \) to a vector is to move the vector in a direction parallel to \( p \).

_T_j. The equation \( Ax = b \) is homogeneous if the zero vector is a solution.
2. a Find the general solutions of the systems $Ax = 0$ whose matrix is:

$$
\begin{bmatrix}
1 & -2 & 3 & -6 & 5 & 0 \\
0 & 0 & 0 & 1 & 4 & -6 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Thus $x_6 = 0/1 = 0$, $x_5$ is free, $x_4 = (0-4x_5-(-6)x_6)/1 = -4x_5$, $x_3$ and $x_2$ are free, and 

$$
x_1 = (0-(-2)x_2-3x_3-(-6)x_4-5x_5)/1 = 2x_2-3x_3+6(-4x_5+6x_6)-5x_5
$$

$= 2x_2-3x_3-29x_5$.

b. Express this solution in parametric vector form.

$$
x = \begin{bmatrix}
2x_2-3x_3-29x_5 \\
x_2 \\
x_3 \\
-4x_5 \\
x_5 \\
0
\end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} x_5.
$$

3. Show that if $Ax = b$ and $Ay = 0$, then for any scalar $\alpha$, $A(x+\alpha y) = b$.

If $Ax = b$ and $Ay = 0$ then for any scalar $\alpha$,

$$
A(x+\alpha y) = Ax + A\alpha y = b + \alpha Ay = b + \alpha 0 = b.
$$

4. Suppose you are to solve $m$ different linear systems of $n$ equations in $n$ unknowns. All of the equations have the same matrix, however, they just differ in right hand sides. Estimate how many multiplications are required.

The elimination step on the matrix requires $\frac{n^3}{3}$ multiplications and each of the $m$

different linear systems requires $n^2$ multiplications for a total of $\frac{n^3}{3} + mn^2$
multiplications.
5. Use the Gaussian Elimination with Partial Pivoting and Solution algorithm to solve
\[3x_1 + 5x_2 - 2x_3 = -16\]
\[-3x_1 - x_3 = -5\]
\[6x_1 + 2x_2 + 4x_3 = 8\]
Show what occupies storage in the \( A \) matrix and the \( ip \) array initially and after each major step of elimination.

\[
\begin{array}{ccc|c}
3 & 5 & -2 & ? \\
-3 & 0 & -1 & ? \\
6 & 2 & 4 & ? \\
\hline
6 & 2 & 4 & 3 \\
-1/2 & 1 & 1 & ? \\
1/2 & 4 & -4 & ? \\
\hline
6 & 2 & 4 & 3 \\
-1/2 & 4 & -4 & 3 \\
1/2 & 1/4 & 2 & ? \\
\hline
6 & 2 & 4 & 3 \\
-1/2 & 4 & -4 & 3 \\
1/2 & 1/4 & 2 & 3 \\
\end{array}
\]

For the elimination applied to \( b = \begin{bmatrix} -16 \\ -5 \\ 8 \end{bmatrix} \) we get it changing to
\[
\begin{bmatrix}
8 \\
-5 \\
-16
\end{bmatrix} \begin{bmatrix}
8 \\
-1 \\
-20
\end{bmatrix} \begin{bmatrix}
8 \\
-20 \\
-1
\end{bmatrix}, \text{ and finally } \begin{bmatrix}
8 \\
-20 \\
4
\end{bmatrix}.
\]
In the bottom loop we get \( b_3 = 4 \) and \( x_3 = \frac{4}{2} = 2 \).

Then we get \( b_2 = -20 - (-4) \cdot 2 = -12 \) so \( x_2 = \frac{-12}{4} = -3 \). Finally we get
\[
b_1 = 8 - 2 \cdot (-3) - 4 \cdot 2 = 6 \text{ so } x_1 = \frac{6}{6} = 1.
\]
6. Fill in the five blanks in the code for Gaussian Elimination with Partial Pivoting and Solution

```plaintext
for k = 1:n
    choose ip_k such that |A_{ip_k,k}| = \max\{|A_{i,k}|; i \geq k\}
    if A_{ip_k,k} = 0
        warning ('Pivot in Gaussian Elimination is zero')
    end
    swap A_{i,k},...,A_{n,k} with A_{ip_k,k},...,A_{ip_k,n}
    for i = k+1:n
        A_{i,k} = A_{i,k} / A_{k,k}
        for j = k+1:n
            A_{i,j} = A_{i,j} - A_{i,k} A_{k,j}
        end
    end
end
for k = 1:n
    swap b_k with b_{ip_k}
    for i = k+1:n
        b_i = b_i - A_{i,k} b_k
    end
end
for i = n:-1:1
    for j = i+1:n
        b_i = b_i - A_{i,j} x_j
    end
    x_i = b_i / A_{i,i}
end
```