M 340L – CS
Homework Set 6 Solutions

1. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of P and A.

   Since $A = PBP^{-1}$, we have $B = P^{-1}(PBP^{-1})P = P^{-1}AP.$

2. Suppose $(B-C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Prove that $B = C.$

   We have $(B-C)D = 0$, so $B - C = (B-C)DD^{-1} = 0D^{-1} = 0$ and $B = C + (B-C) = C + 0 = C.$

3. Suppose A and B are square matrices, B is invertible, and AB is invertible. Prove that A is invertible. [Hint: Let $C = AB$, and solve this equation for A in terms of B and C.]

   If $C = AB$, we have $A = ABB^{-1} = CB^{-1}$, so $A^{-1} = (CB^{-1})^{-1} = (B^{-1})^{-1}C^{-1} = B(AB)^{-1}.$

4. Solve the equation $AB = BC$ for A, assuming that A, B, and C are square and B is invertible.

   We have $A = ABB^{-1} = BCB^{-1}.$

5. Answer true or false to the following. If false offer a counterexample.

a. If u and v are linearly independent, and if w is in $Span\{u,v\}$, then u, v, w are linearly dependent.

   True. If w is in $Span\{u,v\}$ it must be a linear combination of u and v.

b. If three vectors in $\mathbb{R}^3$ lie in the same plane in $\mathbb{R}^3$, then they are linearly dependent.

   False. The three vectors $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are in $\mathbb{R}^3$ lie and lie in the plane $x_3 = 1,$ but are linearly independent.

c. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

False. The set of the single vector $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has two entries but is not linearly independent.

d. If a set in $\mathbb{R}^n$ is linearly dependent then the set contains more than n vectors.

False. The set of the single vector $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ is linearly dependent but does not contains more than 2 vectors.

e. If v_1 and v_2 are in $\mathbb{R}^4$ and v_2 is not a scalar multiple of v_1, then v_1, v_2 are linearly independent.

False. With the vectors $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4$, v_2 is not a scalar multiple of v_1, but v_1, v_2 are linearly dependent since $v_1 = 0v_2$.

f. If v_1, v_2, v_3 are in $\mathbb{R}^3$ and v_3 is not a linear combination of v_1, v_2, then v_1, v_2, v_3 are linearly independent.

False. With the vectors $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$, v_3 is not a linear combination of v_1, v_2, , but v_1, v_2, v_3 are linearly dependent since $v_1 = 0v_2 + 0v_3$.

g. If $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, then $\{v_1, v_2, v_3\}$ is also linearly independent.

True. If no linear combination of the elements of $\{v_1, v_2, v_3, v_4\}$ is zero then no linear combination of the elements of $\{v_1, v_2, v_3\}$ is zero.

6. Answer true or false to the following. If false offer a counterexample.

a. The range of the transformation $x \mapsto Ax$ is the set of all linear combinations of the columns of A.

> **True.** The range of the transformation $x \mapsto Ax$ is the set of all vectors of the form Ax and that equals the set of al linear combinations of the columns of A.

b. Every matrix transformation is a linear transformation.

> **True.** We have $A(\alpha x) = \alpha Ax$ and $A(x+y) = Ax + Ay$ so the transformation $x \mapsto Ax$ is linear.

c. A linear transformation preserves the operations of vector addition and scalar multiplication.

> **True.** We have $T(\alpha x) = A(\alpha x) = \alpha Ax = \alpha T(x)$ and $T(x+y) = A(x+y) = Ax + Ay = T(x) + T(y)$ so the operations of vector addition and scalar multiplication are preserved.

d. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$.

> **True.** Since $T(0) = T(0x) = 0T(x) = 0$ for any vector x, a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$.

7. Answer true or false to the following. If false offer a counterexample.

a. If A is a 4×3 matrix, then the transformation $x \mapsto Ax$ maps $\mathbb{R}^3$ onto $\mathbb{R}^4$.

> **False.** Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then for $x \in \mathbb{R}^3$, Ax has zeros in components 2,3, and 4 and thus for no x is $Ax = b$.

b. Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

> **True.** Let A be an $m\times n$ matrix with columns $T(e_1), \ldots, T(e_n)$ (where $e_1, \ldots, e_n$ are the columns of I_n), then the matrix transformation $x \mapsto Ax$ is equivalent to $x \mapsto T(x)$.

c. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images under T of the columns of the $n \times n$ identity matrix .

> **True.** Let A be an $m \times n$ matrix with columns $T(e_1),...,T(e_n)$ (where $e_1,...,e_n$ are the columns of I_n), then the matrix transformation $x \mapsto Ax$ is equivalent to $x \mapsto T(x)$.

d. A mapping $T:\mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$ (meaning two vectors in $\mathbb{R}^n$ do not map to the same vector in $\mathbb{R}^m$).

> **True.** A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is one-to-one if and only if two vectors in $\mathbb{R}^n$ do not map to the same vector in $\mathbb{R}^m$.

8. Find formulas for X, Y, and Z in terms of I, A, B , and C and inverses. Assume A, B , and C have inverses. (Hint: Compute the product on the left, and set it equal to the right side. First, pretend the blocks are simply real numbers but make sure you do not ever divide – you may multiply by inverses, however. Be careful about right and left multiplication.) In all cases, assume the block matrix dimensions are such that the products are defined.

a. $\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

> We have $XA + 0B = I, YA + ZB = 0, X0 + 0C = 0, \text{and} \, Y0 + ZC = I,$ so
> $XA = I, YA = -ZB,$ and $ZC = I$. We conclude that
> $X = A^{-1}, \mathrm{Z} = \mathrm{C}^{-1}, \text{and} \, Y = -ZBA^{-1} = -C^{-1}BA^{-1}$.

b. $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$

We have
$AX + B0 = I, 0X + I0 = 0, AY + B0 = 0, 0Y + I0 = 0, AZ + BI = 0, \text{and } 0Z + II = I,$ so
$AX = I, AY = 0,$ and $AZ = -B.$ We conclude that $X = A^{-1}, \mathrm{Y} = 0, \text{and } Z = -A^{-1}B.$

c. $\begin{bmatrix} I & 0 & 0 \\ A & I & 0 \\ B & C & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & Z & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$

Omitting the obvious relations, We have
$AI + IX = 0, BI + CX + IY = 0, \text{and } B0 + CI + IZ = 0,$ so
$A + X = 0, B + CX + Y = 0,$ and $C + Z = 0.$ We conclude that
$X = -A, Z = -C, \text{and } Y = -B - CX = CA - B.$