M 340L – CS
Homework Set 4 Solutions

1. Mark T(rue) or F(alse) for each of the following statements:

__T_a. The column space of A is the set of all vectors that can be written as Ax for some x.

__F_b. Elementary row operations on an augmented matrix can change the solution set of the associated linear system.

__T_c. If b is in the set spanned by the columns of A then the equation $Ax = b$ is consistent.

__F_d. For an $n \times n$ system of linear equations, the Gaussian Elimination Algorithm with Partial Pivoting and Elimination Separated from Solving uses approximately $n^3/3$ floating point multiplications and $2n^3/3$ floating point additions/subtractions.

__T_e. the Gaussian Elimination Algorithm with Partial Pivoting has multipliers no larger than one in absolute value.

__T_f. A homogeneous equation is always consistent.

__F_g. The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.

__F_h. If x is a nontrivial solution of $Ax = 0$, then every entry in x is nonzero.

__T_i. The effect of adding p to a vector is to move the vector in a direction parallel to p.

__T_j. The equation $Ax = b$ is homogeneous if the zero vector is a solution.

2. a Find the general solutions of the systems $Ax=0$ whose matrix is:

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_6 = 0/1 = 0$, x_5 is free, $x_4 = (0-4x_5-(-6)x_6)/1 = -4x_5$, x_3 and x_2 are free, and $x_1 = (0-(-2)x_2 - 3x_3 - (-6)x_4 - 5x_5)/1 = 2x_2 - 3x_3 + 6(-4x_5 + 6x_6) - 5x_5$
$= 2x_2 - 3x_3 - 29x_5$.

b. Express this solution in parametric vector form.

$$x = \begin{bmatrix} 2x_2 - 3x_3 - 29x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}.$$

3. Show that if $Ax=b$ and $Ay=0$, then for any scalar α, $A(x+\alpha y)=b$.

If $Ax=b$ and $Ay=0$ then for any scalar α,
$$A(x+\alpha y) = Ax + A\alpha y = b + \alpha Ay = b + \alpha 0 = b.$$

4. Suppose you are to solve *m* different linear systems of *n* equations in *n* unknowns. All of the equations have the same matrix, however, they just differ in right hand sides. Estimate how many multiplications are required.

The elimination step on the matrix requires $\frac{n^3}{3}$ multiplications and each of the *m* different linear systems requires n^2 multiplications for a total of $\frac{n^3}{3} + mn^2$ multiplications.

5. Use the Gaussian Elimination with Partial Pivoting and Solution algorithm to solve

$$3x_1 + 5x_2 - 2x_3 = -16$$
$$-3x_1 \qquad - x_3 = -5$$
$$6x_1 + 2x_2 + 4x_3 = 8$$

Show what occupies storage in the A matrix and the ip array initially and after each major step of elimination.

A ip

3	5	-2		?
-3	0	-1		?
6	2	4		?

6	2	4		3
-1/2	1	1		?
1/2	4	-4		?

6	2	4		3
-1/2	4	-4		3
1/2	1/4	2		?

6	2	4		3
-1/2	4	-4		3
1/2	1/4	2		3

For the elimination applied to $b = \begin{bmatrix} -16 \\ -5 \\ 8 \end{bmatrix}$ we get it changing to

$\begin{bmatrix} 8 \\ -5 \\ -16 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ -20 \end{bmatrix}, \begin{bmatrix} 8 \\ -20 \\ -1 \end{bmatrix}$, and finally $\begin{bmatrix} 8 \\ -20 \\ 4 \end{bmatrix}$. In the bottom loop we $b_3 = 4$ and $x_3 = \frac{4}{2} = 2$.

Then we get $b_2 = -20 - (-4) \cdot 2 = -12$ so $x_2 = \frac{-12}{4} = -3$. Finally we get

$b_1 = 8 - 2 \cdot (-3) - 4 \cdot 2 = 6$ so $x_1 = \frac{6}{6} = 1$.

6. Fill in the five blanks in the code for Gaussian Elimination with Partial Pivoting and Solution

for *k* = 1:*n*

  choose ip_k such that $|A_{ip_k,k}| = \max\{|A_{i,k}| : i \geq k\}$

  if $A_{ip_k,k}$ = 0

    warning ('Pivot in Gaussian Elimination is zero')

  end

  swap $A_{k,k},\ldots,A_{k,n}$ with $A_{ip_k,k},\ldots,A_{ip_k,n}$

  for *i* = <u>*k+1*</u>:*n*

    $A_{i,k}$ = $\underline{A_{i,k} / A_{k,k}}$

    for *j* = *k*+1:*n*

      $A_{i,j} = \underline{A_{i,j} - A_{i,k} A_{k,j}}$

    end

  end

end

for *k* = 1:*n*

  swap b_k with $\underline{b_{ip_k}}$

  for *i* = *k*+1:*n*

    $b_i = b_i - A_{i,k} b_k$

  end

end

for *i* = *n*:-1:1

  for *j* = *i*+1:*n*

    $b_i = b_i - A_{i,j} x_j$

  end

  $x_i = \underline{b_i / A_{i,i}}$

end