**M 340L – CS**
**Homework Set 2 Solutions**

Find the general solutions of the systems whose augmented matrices are given in Problems 1 - 4.

1.

$$\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & -2 & 0 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$x_3$ is free, $x_2 = 3$, $x_1 = 4$.

2.

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & -7 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$x_3 = -2$, $x_2$ is free. $x_1 = 2-(-2)x_2 = 2+2x_2$.

3.

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2/3 & 4/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3$ is free, $x_2$ is free. $x_1 = 0-(-2/3)x_2 - 4/3 \cdot x_3 = 2/3x_2 - 4/3x_3$.

4.

$$\begin{bmatrix} 1 & 0 & 5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_5 = 0$, $x_4$ is free, $x_3$ is free. $x_2 = 6 - 4 \cdot x_3 - (-1) \cdot x_4 = 6 - 4x_3 + x_4, x_1 = 3 - 5 \cdot x_3$.

5. Key statements from Lay, Section 1.2 are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, cite an example that shows the statement is not true in all cases or give the location of a statement that has been quoted or used incorrectly.)

a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

> **False.** "Each matrix is row equivalent to one and only one reduced echelon matrix."

b. The row reduction algorithm applies only to augmented matrices for a linear system.

> **False.** "The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system."

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

> **True.** "The variables ... corresponding to pivot columns in the matrix are called basic variables."

d. Finding a parametric description of the solution set of a linear system is the same as solving the system.

> **True.** "Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty."

e. If one row in an echelon form of an augmented matrix is $[\,0\ 0\ 0\ 5\ 0]$, then the associated linear system is inconsistent.

**False.** "A linear system is consistent if and only if the rightmost column of the augmented
matrix is not a pivot column— that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \ \cdots \ 0 \ \ b]$ with b nonzero."

6. Write a system of equations that is equivalent to the vector equation:

$$x_1\begin{bmatrix}3\\-2\end{bmatrix}+x_2\begin{bmatrix}7\\3\end{bmatrix}+x_3\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

$$\begin{aligned}3x_1+7x_2-2x_3&=0\\-2x_1+3x_2+x_3&=0\end{aligned}$$

7. Write a vector equation that is equivalent to system of equations:

$$\begin{aligned}3x_1-2x_2+4x_3&=3\\-2x_1-7x_2+5x_3&=1\\5x_1+4x_2-3x_3&=2\end{aligned}$$

$$x_1\begin{bmatrix}3\\-2\\5\end{bmatrix}+x_2\begin{bmatrix}-2\\-7\\4\end{bmatrix}+x_3\begin{bmatrix}4\\5\\-3\end{bmatrix}=\begin{bmatrix}3\\1\\2\end{bmatrix}.$$

8. Let $A=\begin{bmatrix}2&0&6\\-1&8&5\\1&-2&1\end{bmatrix}$, let $\boldsymbol{b}=\begin{bmatrix}10\\3\\7\end{bmatrix}$, and let and let $W$ be the set of all linear combinations of the columns of $A$.

a. Determine if $b$ is in $W$?

$$\begin{bmatrix}2&0&6&10\\-1&8&5&3\\1&-2&1&7\end{bmatrix}\rightarrow\begin{bmatrix}2&0&6&10\\0&8&8&8\\0&-2&-2&2\end{bmatrix}\rightarrow\begin{bmatrix}2&0&6&10\\0&8&8&8\\0&0&0&4\end{bmatrix}$$

The system is inconsistent: there is no solution. $b$ is not in $W$.

b. Show that the second column of $A$ is in $W$.

For $x_1 = 0, x_2 = 1, x_3 = 0$, we have $x_1\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$.

9. Compute the product:

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

10. Write the matrix equation as a vector equation:

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

$$-3\begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

11. Write the vector equation as a matrix equation:

$$z_1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2\begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3\begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

12. Given $A$ and $b$, write the augmented matrix for the linear system that corresponds to the matrix equation. $Ax=b$. Then solve the system and write the solution as a vector.

$$A=\begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}, b=\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 4 & 12 \end{bmatrix},$$

$$x_3=\frac{12}{4}=3, x_2=\frac{5-(-1)\cdot 3}{2}=4, x_1=\frac{1-2\cdot 4-(-1)3}{1}=-4 \qquad ,$$

$$x=\begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

13. Key statements from Lay, Section 1.4 are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.)

a. If the equation $Ax=b$ is consistent, then $b$ is in the set spanned by the columns of $A$.

> True. The equation $Ax=b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

b. The solution set of a linear system whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ is the same as the solution set of $Ax=b$, if $A=[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$.

> True. If $A$ is an $m\times n$ matrix, with columns $\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n$, and if $b$ is in $\mathbb{R}^m$, the matrix equation $Ax=b$ has the same solution set as the vector equation $x_1\mathbf{a}_1+x_2\mathbf{a}_2+\cdots+x_n\mathbf{a}_n=b$ which, in turn, has the same solution set as the system of linear equations whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad b]$.