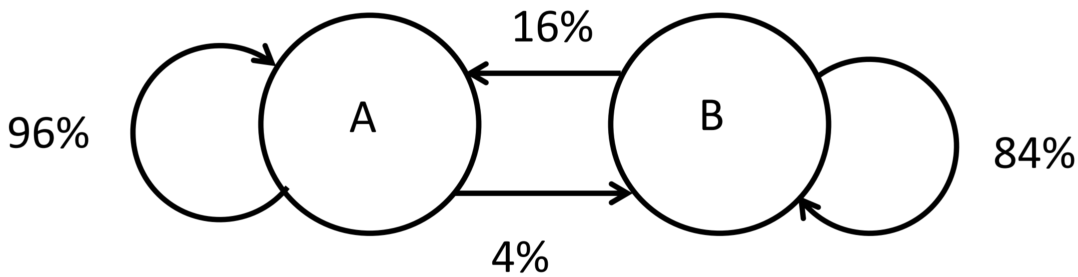
M 340L – CS
Homework Set 12

Note: Scale all eigenvectors so the largest component is + 1.

1. A Markov process:

A city has two restaurants: A and B. 96% of the time, a person leaving A will return to A the next time she goes to either A or B. Thus, 4% of the time, she will switch to B the next time. 84% of the time, a person leaving B will return to B the next time she goes to either A or B. Thus, 16% of the time, she will switch to A the next time. This diagram summarizes the situation:



Let $A=\begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix}$ (i.e. the matrix of transitions). use $y=\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

a. What are its eigenvalues and eigenvectors?

The characte3ristic polynomial is $(24/25-\lambda)(21/25-\lambda)-4/25^2=\lambda^2-9/5\lambda+4/5=(\lambda-1)(\lambda-4/5)$ so the eigenvalues are 1 and 4/5. The null space of $A-I=\begin{bmatrix} -1/25 & 4/25 \\ 1/25 & -4/25 \end{bmatrix}$. is the vector $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$(and its multiples). The null space of $A-4/5I=\begin{bmatrix} 4/25 & 4/25 \\ 1/25 & 1/25 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$(and its multiples). Thus the eigenvectors are $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. Using part a. express $A = VDV^{-1}$ (where the columns of V are the eigenvectors and D is a diagonal matrix containing the associated eigenvalues.)

Since $\begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}$,

$$A = \begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix}.$$

c. Using the fact that $A^k = VD^kV^{-1}$, what is $A^{100}y$, for $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$? (See comments in 1c.)

$$A^{100}y = VD^{100}V^{-1}y = \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix} \begin{bmatrix} 4/5 & 4/5 \\ 1/5 & -4/5 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & (4/5)^{100} \end{bmatrix} \begin{bmatrix} 4/5 \\ -3/10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & -1 \end{bmatrix} \begin{bmatrix} 4/5 \\ -3 \cdot (4/5)^{100}/10 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 - 3 \cdot (4/5)^{100}/10 \\ 1/5 + 3 \cdot (4/5)^{100}/10 \end{bmatrix}$$

d. Express your answer in part **c** as $A^{100}y = \gamma\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is + 1. Compare $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ to the eigenvector corresponding to λ_1.

$$A^{100}y == \begin{bmatrix}4/5-3\cdot(4/5)^{100}/10\\ 1/5+3\cdot(4/5)^{100}/10\end{bmatrix} = (4/5-3\cdot(4/5)^{100}/10)\begin{bmatrix}1\\ \dfrac{1+15\cdot(4/5)^{100}/10}{4-15\cdot(4/5)^{100}/10}\end{bmatrix}.$$ The vector $\begin{bmatrix}1\\ \dfrac{1+15\cdot(4/5)^{100}/10}{4-15\cdot(4/5)^{100}/10}\end{bmatrix}$ is very close (within 10^{-10}) to the eigenvector $\begin{bmatrix}1\\ 1/4\end{bmatrix}$.

e. Using part **b.**, what is A^{100}?

$$A^{100} = VD^{100}V^{-1} = \begin{bmatrix}1 & 1\\ 1/4 & -1\end{bmatrix}\begin{bmatrix}1^{100} & 0\\ 0 & (4/5)^{100}\end{bmatrix}\begin{bmatrix}4/5 & 4/5\\ 1/5 & -4/5\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\ 1/4 & -1\end{bmatrix}\begin{bmatrix}4/5 & 4/5\\ (4/5)^{100}/5 & -(4/5)^{101}\end{bmatrix}$$
$$= \begin{bmatrix}4/5+(4/5)^{100}/5 & 4/5-(4/5)^{101}\\ 1/5-(4/5)^{100}/5 & 1/5+(4/5)^{101}\end{bmatrix}.$$