M 340L – CS
Homework Set 12 Solutions

Note: Scale all eigenvectors so the largest component is + 1.

1. How do perturbations affect eigenvalues and eigenvectors?

a. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (See the note above regarding the scaling of eigenvectors and make sure you do it throughout the homework.)

The eigenvalues are 2 and 2. The null space of $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and its multiples). $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is only one linearly independent eigenvector.

b. Let $B = \begin{bmatrix} 2 & 1 \\ 0 & 2+\varepsilon \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Your answers should be in terms of the perturbation parameter ε.)

The eigenvalues are 2 and $2+\varepsilon$. The null space of $B - 2I = \begin{bmatrix} 0 & 1 \\ 0 & \varepsilon \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and its multiples). The null space of $B - (2+\varepsilon)I = \begin{bmatrix} -\varepsilon & 1 \\ 0 & 0 \end{bmatrix}$ is the vector $\begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}$ (and its multiples).

Thus the eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}$.

c. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of A. Comment on the linear independence of the eigenvectors of B.

The perturbation has introduced a second eigenvector but it is nearly linearly dependent upon the first.

d. Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Choose linearly independent eigenvectors.)

The eigenvalues are 2 and 2. The null space of $C-2I=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is all of $\mathbb{R}^2$. Two linearly independent eigenvectors are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

e. Let $D=\begin{bmatrix}2&\varepsilon\\0&2\end{bmatrix}$. What are its eigenvalues and eigenvectors?

The eigenvalues are 2 and 2. The null space of $D-2I=\begin{bmatrix}0&\varepsilon\\0&0\end{bmatrix}$ is the vector $\begin{bmatrix}1\\0\end{bmatrix}$ (and its multiples). The only eigenvector is $\begin{bmatrix}1\\0\end{bmatrix}$.

f. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of C.

The perturbation has left the eigenvalues unperturbed but has removed the second eigenvector.

2. Using the diagonal form to compute high powers:

Let $A=\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$. *Feel free to express answers in parts* **c**, **d**, *and* **e** *using expressions involving powers.*

a. What are its eigenvalues and eigenvectors?

The characteristic polynomial is $(1-\lambda)(1-\lambda)-4=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)$ so the eigenvalues are 3 and -1. The null space of $A-3I=\begin{bmatrix}-2&-2\\-2&-2\end{bmatrix}$ is the vector $\begin{bmatrix}1\\-1\end{bmatrix}$ (and its multiples). The null space of $A-(-1)I=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$ is the vector $\begin{bmatrix}1\\1\end{bmatrix}$ (and its multiples). Thus the eigenvectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$.

b. Using part **a.**, form D, a diagonal matrix of eigenvalues, form V whose columns are the associated eigenvectors, then compute V^{-1}, and finally VDV^{-1}. Compare VDV^{-1} to A.

Since $\begin{bmatrix}1&1\\-1&1\end{bmatrix}^{-1}=\begin{bmatrix}1/2&-1/2\\1/2&1/2\end{bmatrix}$, $A=\begin{bmatrix}1&-2\\-2&1\end{bmatrix}=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}1/2&-1/2\\1/2&1/2\end{bmatrix}$.

c. Using part **b.**, what is $A^{100}y$, for $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$? (Do **not** compute A^{100} - yet. Use associativity in a clever way.)

$$A^{100}y = VD^{100}V^{-1}y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix}\begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -3^{100}/2 \\ 3/2 \end{bmatrix}$$
$$= \begin{bmatrix} (3-3^{100})/2 \\ (3^{100}+3)/2 \end{bmatrix}$$

d. Express your answer in part **c** as $A^{100}y = \gamma\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is + 1. Compare $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to the eigenvector corresponding to λ_1.

$A^{100}y = \begin{bmatrix} (3-3^{100})/2 \\ (3^{100}+3)/2 \end{bmatrix} = \dfrac{(3^{100}+3)}{2}\begin{bmatrix} \dfrac{-3^{100}+3}{3^{100}+3} \\ 1 \end{bmatrix}$. The vector $\begin{bmatrix} \dfrac{-3^{100}+3}{3^{100}+3} \\ 1 \end{bmatrix}$ is very close (within 10^{-46}) to the negative of the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

e. Using part **b.**, what is A^{100}?

$$A^{100} = VD^{100}V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3^{100}/2 & -3^{100}/2 \\ 1/2 & 1/2 \end{bmatrix}$$
$$= \begin{bmatrix} (3^{100}+1)/2 & (-3^{100}+1)/2 \\ (-3^{100}+1)/2 & (3^{100}+1)/2 \end{bmatrix}.$$

3. All zero eigenvalues:

Find a simple non-zero matrix having all zero eigenvalues.

The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a non-zero matrix having all zero eigenvalues