Workshop III - SDP Workshop

September 16, 2016

1 Max-2-SAT

Recall that in an instance of Max-2-SAT we have a set of n $\{-1,1\}$ variables, and m constraints of the form

$$C_i \equiv b_{i1}x_{i1} + b_{i2}x_{i2} \neq -2$$

where the x_{ij} are variables and the $b_{ij} \in \{-1,1\}$ are signs fixed in the instance. The objective is satisfy as many constraints as possible. The trivial algorithm of choosing a random assignment achieves a $\frac{3}{4}$ approximation. Adapt the approximation algorithm we saw in class for Max-Cut to achieve an α approximation algorithm for some constant $\alpha > 0.75$.

2 Unique Games Approximation

Recall that an instance of Unique Label cover consists of a bipartite graph (L, R, E), a finite alphabet of labels Σ, and a set of permutations $\{f_e : \Sigma \to \Sigma \mid e \in E\}$. A labelling is a function $l : L \cup R \to \Sigma$. An edge $e = (u, v)$ is satisfied if $f_e(l(u)) = l(v)$. Our objective is to find an l so as to maximize the fraction of edges satisfied. Give an SDP+rounding based algorithm which given an instance which is $1 - \epsilon$ satisfiable (that is some assignment satisfies $1 - \epsilon$ fraction of edges) can expect to satisfy $1 - poly(k, 1/\epsilon)$ fraction of edges.

3 Lovász Sandwich Theorem

Let $G = (V, E)$ be an undirected graph. The theta function of $\overline{G}$ (of the complement of G that is), $\vartheta(\overline{G})$ is defined as the optimum of the following semidefinite program:

$$\begin{aligned} \underset{X}{\text{maximize}} \quad & \sum_{i,j} X_{i,j} \\ \text{subject to} \quad & X_{i,j} = 0,\ (i,j) \notin E, \\ & Tr(X) = 1,\ X \succeq 0. \end{aligned}$$

Show that the above program is in fact a relaxation for the Max-Clique problem. Conclude that $\omega(G) \leq \vartheta(\overline{G})$.

Let J be the all ones matrix. An equivalent (you don't need to prove this) definition for $\vartheta(\overline{G})$ is

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & k \\ \text{subject to} \quad & Y_{i,j} = 0,\ (i,j) \in E, \\ & Y_{i,i} = 0, \\ & kI - J - Y \succeq 0. \end{aligned}$$

Show that the above program is in fact a relaxation for the minimum coloring problem. Conclude that $\vartheta(\overline{G}) \leq \chi(G)$.

4 Integrality Gaps for Label Cover SDP Relaxation

Consider the following SDP relaxation for the label cover problem on a bipartite graph (L, R, E) with labels coming from an alphabet $[k]$, and edge constraints f_e:

$$\begin{aligned}
\underset{X}{\text{maximize}} \quad & \sum_{(u,v)\in E} \sum_{f_{(u,v)}(i)=j} X_{(u,i),(v,j)} \\
\text{subject to} \quad & \sum_{i\in[k]} X_{(u,i),(u,i)} = 1,\ u \in V, \\
& X_{(u,i),(u,j)} = 0,\ u \in V\ i,j \in [k]\ i \neq j, \\
& X \succeq 0.
\end{aligned}$$

Show that the above program is in fact a relaxation for the label cover problem. Then construct an infinite family of label cover instances such that for k which grows sufficiently fast with $|E|$ the SDP optimum is $|E|$, but the true integral optimum is $O(\sqrt{|E|})$.