Workshop IV - $\sum$um of S$\square$uares Workshop

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1 Pseudo-distributions

Recall that we can view the solution returned by d levels of the Sum of Squares hierarchy as a function $\mu : \{0,1\}^n \to \mathbb{R}$ such that there exists a formal "expectation" operator $\widetilde{\mathbb{E}}_{\mu}$ which acts on functions as follows:

$$\widetilde{\mathbb{E}}_{\mu} f = \sum_{x \in \{0,1\}^n} \mu(x) f(x)$$

We say that μ is a degree d pseudo-distribution if the above operator, called a pseudo-expectation, satisfies

$$\widetilde{\mathbb{E}}_{\mu} 1 = 1$$

and for all polynomials p of degree at most $d/2$

$$\widetilde{\mathbb{E}}_{\mu} p^2 \geq 0$$

Show that pseudo-expectation satisfies a version of Cauchy-Schwarz. In particular if μ is a degree d pseudo-distribution show that for polynomials p, q of degree at most $d/2$,

$$(\widetilde{\mathbb{E}}_{\mu} pq)^2 \leq (\widetilde{\mathbb{E}}_{\mu} p^2)(\widetilde{\mathbb{E}}_{\mu} q^2)$$

2 3 XOR Lower Bound

Suppose we are given a system of m equations over n variables in $\{0,1\}$ of the following form:

$$x_{i1} + x_{i2} + x_{i3} = a_i \text{ (mod 2)}$$

where $a_i \in \{0,1\}$. In this exercise we will show that for all $\epsilon > 0$ there exists such a system of equations such that every assignment $x \in \{0,1\}^n$ satisfies at most $1/2 + \epsilon$ fraction of equations, but the optimal fraction of equations "satisfied" by $\Omega(n)$ levels of the Sum of Squares hierarchy is 1.

It will be convenient to view our system of equations as a 3-left-regular bipartite graph $G = (L \cap R, E)$ where there is a vertex in L for each equation, a vertex in R for each variable, and we join each vertex v on the left to the three vertices on the right which correspond to the variables in the equation corresponding to v. Finally, we can write down the a_is as a vector $a \in \{0,1\}^m$.

First we show that a randomly selected system is with high probability not much more than 1/2 satisfiable.

2.1 Presentation 1 - Soundness

Fix $\epsilon > 0$, and let n be the number of variables. Suppose our system has $m > 9n/\epsilon^2$ equations. Show that if we select an $a \in \{0,1\}^m$ uniformly at random, with probability $1 - o_n(1)$ no assignment $x \in \{0,1\}^n$ satisfies more that $1/2 + \epsilon$ fraction of equations.

Let G be a bipartite graph which is left d-regular. For a set of left vertices S we denote its set of neighbors $\Gamma(S)$. Suppose that for any subset S of left vertices of size at most s, that $\Gamma(S) \geq \alpha|S|$. We call such a graph a (d, s, α) expander.

2.2 Presentation 2 - Expansion of Random Instances

Consider the following probabilistic construction of a system of equations. Independently for each triple x_i, x_j, x_k, we include them together in an equation with probability p/n^2 where p is some number which depends on ϵ but not n. Show that for some choice of constant $\gamma \in [0,1]$, that a graph sampled in this manner is a $(3, \gamma n, 1.7)$ expander with probability at least 0.9.

The upshot of the previous two exercises is that a randomly chosen set of equations is not very satisfiable, and its induced graph is an expander. Now, we will construct a pseudo-distribution μ of degree $d = \gamma n/10$ which "satisfies" all equations.

Given an equation $x_{i1} + x_{i2} + x_{i3} = a_i$ we can encode it as a polynomial $(1 - 2a_i)(1 - 2x_{i1})(1 - 2x_{i2})(1 - 2x_{i3})$, which evaluates to 1 precisely when the equation is satisfied. Let $\chi_S = \prod_{i \in S}(1 - 2x_i)$ for $S \subset [n]$, and let $\chi_\emptyset$ be identically 1. Now, the space of polynomials (with boolean inputs) of degree $\leq d$ is spanned by polynomials $\{\chi_S \ : \ |S| \leq d\}$, so to construct a degree d pseudo-distribution μ we need only specify the value of $\underset{\mu}{\widetilde{\mathbb{E}}}$ on these polynomials.

2.3 Presentation 3 - Pseudo-Distribution Construction and Local Consistency

We will construct the pseudo-distribution μ in the following way. First, we set

$$\underset{\mu}{\widetilde{\mathbb{E}}} \chi_\emptyset = 1$$

Next, since it must satisfy $(1-2a_i)(1-2x_{i1})(1-2x_{i2})(1-2x_{i3})$ for each equation $x_{i1}+x_{i2}+x_{i3}=a_i$, we set

$$\widetilde{\mathbb{E}}_{\mu}(1-2x_{i1})(1-2x_{i2})(1-2x_{i3}) = (1-2a_i)$$

Now, we continue in the following manner as long as possible: Pick a pair of subsets with $|S|, |T| \leq d$ such that $\widetilde{\mathbb{E}}_{\mu}\chi_S$ and $\widetilde{\mathbb{E}}_{\mu}\chi_T$ have been set. If it hasn't been set yet and $|S \triangle T| \leq d$, assign

$$\widetilde{\mathbb{E}}_{\mu}\chi_{S\triangle T} = (\widetilde{\mathbb{E}}_{\mu}\chi_S)(\widetilde{\mathbb{E}}_{\mu}\chi_T)$$

Else if the left hand side has been set to a value other than the right hand side this process ends and fails. Finally if we can no longer continue this process, for all unassigned subsets S of size at most d we set

$$\widetilde{\mathbb{E}}_{\mu}\chi_S = 0$$

Show that if the graph associated with our system of equations is a $(3, 10n, 1.7)$ expander, that the above process never fails.

2.4 Presentation 4 - Pseudo-Distribution is Positive Semidefinite

The last step is to show that we've constructed μ as a valid pseudo-distribution. In particular prove that for a polynomial p with degree at most $d/2$,

$$\widetilde{\mathbb{E}}_{\mu}p^2 \geq 0$$

. Conclude that after $\Omega(n)$ rounds of Sum of Squares heirarchy the SDP has an integrality gap of 2. In particular we've shown that Sum of Squares cannot outperform the trivial algorithm of a random assignment in polynomial time.