# Semidefinite Programming

**Max-Cut:** Given $G=(V,E)$ find a cut containing as many edges as possible.

## 1 Random assignment / greedy algorithm:

$1/2$ - approximation.

## 2 Linear programming

are i,j in different parts?

$$\max \sum_{(i,j)\in E} d_{ij}$$

$$d_{ij}+d_{jk}+d_{ki} \leq 2 \qquad \forall i,j,k$$

$$d_{ij}+d_{jk} \geq d_{ik}$$

$$1 \geq d_{ij} \geq 0 \qquad \forall i,j$$


Lem Integrality gap $= \frac{1}{2}$

Today A stronger technique!

Max-cut is captured by a quadratic program:

$$\max \frac{|E|}{2} - \frac{1}{2}\sum_{(i,j)\in E} v_i v_j$$

s.t. is $v_i$ in 1'st part?

$$v_i \in \{\pm 1\}$$

<u>Idea</u> Let $v_i$'s be vectors in $\mathbb{R}^n$.

Relaxation:

$$\max \frac{|E|}{2} - \frac{1}{2} \cdot \sum_{(i,j)\in E} \langle v_i, v_j \rangle$$

$$\text{s.t.} \quad \|v_i\|^2 = 1$$

This is an instance of "semidefinite programming".

It turns out that we can (approx) solve such programs in polynomial time.

<u>Thm</u> (positive semidefinite matrix)

The following are equivalent for an $n \times n$ matrix $P$:

(1) $\exists$ $m \times n$ matrix $V$ s.t. $P = V^T V$ ← Note: can efficiently compute $V$ from $P$ (Cholesky decomposition)

(2) $\forall\ x \in \mathbb{R}^n \quad x^T P x \geq 0$

(3) All eigenvalues of $P$ are non-negative.

When these conditions hold we say that $P$ is positive semidefinite and denote $P \succcurlyeq 0$.

Note that (1) is the property we needed.
(2) shows that this property is equivalent to <u>linear constraints</u>.

Issue: there are infinitely many constraint.
Solution: efficient separation oracle using (3):
Eigenvalues of $P$ can be computed efficiently so we can check whether (3) holds.
If not, eigenvector corresponding to negative eigenvalue gives unsat constraint.

Comments 1 eigenvectors / eigenvalues / solutions
to SDP may not have poly-size representations!

However -

- If feasible region is nonempty & contained in
a poly size ball centered at origin.

we can get an approximation up to additive $\varepsilon$.
arbitrarily small

2 SDP solver only guarantees vectors in
some dimension. Wlog this dimension is $\leq n$.

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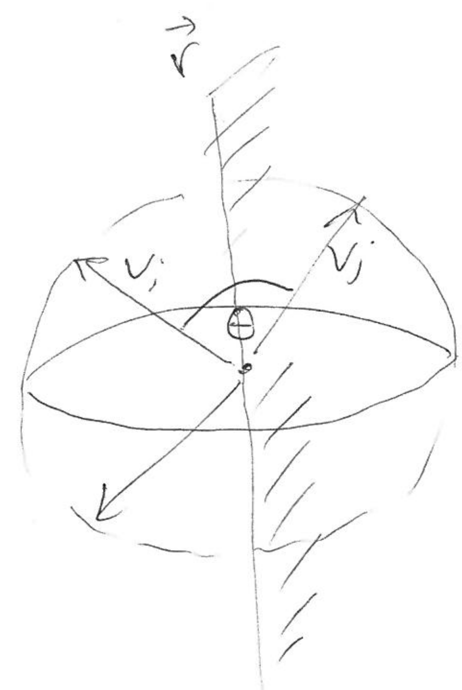
# Back to Max-Cut

Solving relaxation yields $\vec{v}_i \in \mathbb{R}^n$ $1 \le i \le n$

s.t. $\frac{|E|}{2} - \frac{1}{2} \sum_{(i,j) \in E} \langle \vec{v}_i, \vec{v}_j \rangle \geq$ max # edges cut.

$\|\vec{v}_i\|^2 = 1$




Recall: $\langle v_i, v_j \rangle = \cos \measuredangle(v_i, v_j)$

$\langle v_i, v_j \rangle = 0 \rightarrow \cos \measuredangle = 0$

$\langle v_i, v_j \rangle = 1 \rightarrow \cos \measuredangle = 1$

$\langle v_i, v_j \rangle = -1 \rightarrow \cos \measuredangle = -1$

## Randomized rounding

Pick a random $\vec{r} \in \mathbb{R}^n$ $\|\vec{r}\|^2 = 1$

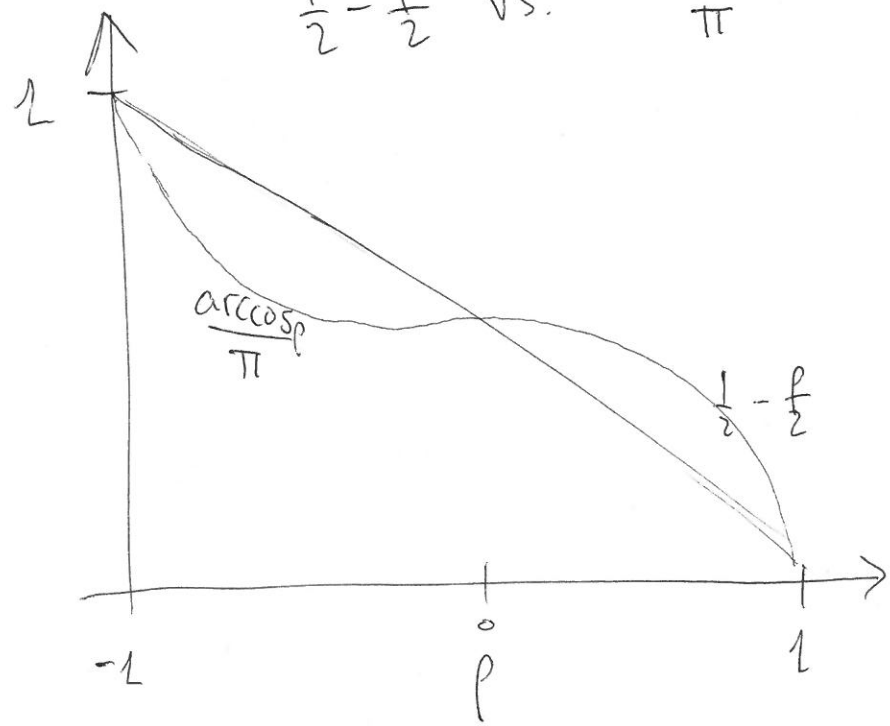
set assignment to $i$ to be $\operatorname{sign} \langle \vec{r}, \vec{v}_i \rangle$.

$$\mathbb{E}\left[\#\text{edges cut}\right] = \mathbb{E}_{(i,j) \in E} \, p(ij \text{ cut}) = \frac{\measuredangle(\vec{v}_i, \vec{v}_j)}{\pi} =$$

$$\frac{\arccos \langle \vec{v}_i, \vec{v}_j \rangle}{\pi}$$

OPT vs. Rounded

$\frac{1}{2} - \frac{\rho}{2}$ vs. $\frac{\arccos \rho}{\pi}$




The largest gap is $\min \dfrac{\frac{\arccos \rho}{\pi}}{\frac{1}{2} - \frac{\rho}{2}} \approx 0.878.$

Therefore, get an approx factor of for Max-Cut.

# Integrality Gap

It turns out that the Goemans-Williamson rounding procedure is optimal!

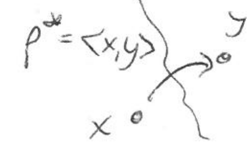
Consider the graph whose vertices are points $\vec{v} \in \mathbb{R}^n$ on the unit sphere,
and where edges connect $\vec{v}$ and $\vec{u}$ iff $\langle \vec{v}, \vec{u} \rangle = \rho^*$ where $\rho^*$ minimizes $\dfrac{\frac{\arccos\rho}{\pi}}{\frac{1}{2} - \frac{\rho}{2}}$

(graph should be suitably discretized).

By construction SDP value $= \frac{1}{2} - \frac{\rho^*}{2}$.
We claim that no cut has value better than $\approx \frac{\arccos\rho}{\pi}$.

Cut = set on unit sphere

$\#\text{edges}_{\text{cut}}$ = boundary of set w/ steps




Thm [Feige-Schechtman] best cut is a hemisphere (cap through origin)

I.e. GW rounding is optimal. It achieves $\frac{\arccos\rho}{\pi}$.