Workshop II - Local Search, Primal Dual, Sherali Adams

The workshop will be held on **Monday, September 17th**. Each student must submit a written solution to 1 problem of their choice **at the beginning of class on the day of the workshop**. Specify which problem you chose at the top of your submission. Alternatively, a student may volunteer to present a problem to the class in the workshop. The workshop will have 3 presenters, each presenting one problem in the order given here. Each presentation should be about 5 minutes. Register to present on September 12th, upon an announcement from the TA.

1 Local Optima for Facility Location

Recall the metric uncapacitated facility location problem as discussed in class. In this problem, we have a set $\mathcal{F}$ of facilities and a set D of clients. There is a facility cost $f_i \in \mathbb{R}^+$ for every $i \in \mathcal{F}$. Moreover, for every $i, j \in \mathcal{F} \cup D$ there is an associated distance c_{ij} that satisfies the triangle inequality. The goal is find a set $S \subset \mathcal{F}$ of facilities and an assignment of clients to opened facilities $\sigma : D \to S$ such that we minimize

$$\sum_{i \in S} f_i + \sum_{j \in D} c_{j,\sigma(j)}$$

For a given instance of the facility location problem with optimal solution S^* and σ^*, define $F^* = \sum_{i \in S^*} f_i$ and $C^* = \sum_{j \in D} c_{j,\sigma^*(j)}$

We call a solution S, σ *locally optimal* if the following "updates" to the solution do not improve the value objective function:

- **Add**: add a new facility i to S so that the new solution is $S \cup \{i\}$, reassign all clients as efficiently as possible
- **Remove**: remove a facility from S, reassign all clients as efficiently as possible
- **Swap**: remove an old facility from S and add a new facility to S, reassign all clients as efficiently as possible.

1.1 Bound on Assignment Cost

Let S, σ be a locally optimal solution, and let $C = \sum_{j \in D} c_{j,\sigma(j)}$. Prove that $C \leq F^* + C^*$.

Now let $F = \sum_{j \in D} c_{j,\sigma(j)}$ and assume that $F \leq F^* + 2C^*$ (This would take longer to prove, so you don't need to do this). Given this fact and what you just proved, suggest a natural "greedy" 3-approximation algorithm for the facility location problem.

1.2 Finding a Polynomial Time Algorithm.

Explain why the natural greedy algorithm might NOT be polynomial time. Suggest a way to modify the algorithm so that it is polynomial time with approximation factor $3 + \epsilon$ for arbitrary $\epsilon > 0$.

Hint: Have your algorithm only update the solution if there is a move that improves the solution by a constant factor (say by a factor $(1 - \delta)$). Adjust the argument from problem 1.1 to show that $C - |\mathcal{F}|\delta(F + C) \leq F^* + C^*$. Then, similarly assume (without proof) that we can similarly say $F - |\mathcal{F}|\delta(F + C) \leq F^* + 2C^*$. Combine these two facts and pick δ wisely.

2 Optimal Primal-Dual Algorithm (Harder)

(This is problem 7.5 from the Williamson and Shmoys textbook) In the *minimum cost branching problem* we are given a directed graph $G = (V, A)$, a root vertex $r \in V$, and weights $w_{ij} \geq 0$ for all $(i, j) \in A$. The goal of the problem is to find a minimum cost set of arcs $F \subset A$ such that for every $v \in V$, there is exactly one directed path in F from r to v. Use the primal-dual method to give an **optimal** algorithm to this problem.

Note this problem will require you to formulate your own LP relaxation of the problem, consider its dual, and devise algorithm based on the dual LP.

3 Proof Complexity and Integrality Gaps for Sherali-Adams (Longer)

NOTE: This problem has many definitions in it, be sure to read all of them carefully in order to understand what is asked of you.

In the **k-XOR** problem we are given n boolean variables $x_1 \ldots x_n$ a set of m XOR equations of the form

$$\bigoplus_{i \in I} x_i = b$$

Where $I \subset [n]$, $|I| \leq k$. Let $I_j \subset [n]$ for $1 \leq j \leq m$ be the set of variables associated with the j-th XOR equation. The goal is to find an assignment of values for x_i in $\{0, 1\}$ that maximizes the number of XOR equations satisfied.

The **Sherali-Adams** hierarchy for the k-XOR problem is a sequence of linear programs indexed by $t \in \mathbb{N}$. The t-th level LP in the hierarchy uses variables $X_{S,\alpha}$ where $S \subset [n], |S| \leq t$ and $\alpha \in \{0, 1\}^{|S|}$. The variable $X_{S,\alpha}$ can be thought of as a boolean indicator denoting whether the subset S of variables x_i was assigned values α. The linear program on these variables is defined as:

$$\max \sum_{j=1}^{m} \sum_{\alpha \in \{0,1\}^{|I_j|}} X_{(I_j, \alpha)} \cdot \mathbf{1}\left[\bigoplus_{i \in I_j} \alpha_i = b\right]$$

over the constraints:

$$\forall |S| < t, \alpha \in \{0, 1\}^{|S|}, j \notin S:$$

$$X_{S \cup \{j\}, \alpha \circ 0} + X_{S \cup \{j\}, \alpha \circ 1} = X_{S,\alpha}$$

$$X_{\emptyset,\emptyset} = 1$$

The **width-w resolution** of a set A of XOR equations is the set of all possible XOR equations that can be obtained through repeatedly performing the following operation on the current set of XOR equations: for any $I, J \subset [n]$ with $|I \Delta J| \leq w$ if $\bigoplus_{i \in I} x_i = b$ and $\bigoplus_{j \in J} x_j = b'$ are both in A, then add $\bigoplus_{i \in I \Delta J} x_i = b \oplus b'$ to A.

The following is a rough statement of an important theorem about resolution:

Theorem (rough): Let $k \geq 3$ and consider a randomly selected instance of the k-XOR problem on m equations. Then for some $w = O(n)$, with probability $1 - o(1)$, the width-w resolution of the set of m equations will not contain the equation $0 = 1$, $x_i = 0$ or $x_i = 1$ for any i.

Consider also the following fact about random k-XOR instances (also roughly stated)

Theorem: Consider a random instance of the k-XOR problem on m equations. Then with probability $1 - o(1)$ at most half of the equations can be satisfied simultaneously.

Given these theorems, prove that the t-th level Sherali-Adams LP for the k-XOR problem has an integrality gap of $1/2$ for some $t = O(n)$.