Debts from last week

① "Needle in the haystack" argument:

Fix a tester V. Pick $\vec{x}_0 \in \mathbb{F}^m$ uniformly at random.

Let $p \in \mathbb{F}_{\le d}[x_1, \ldots, x_m]$, $f_{\vec{x}_0} : \mathbb{F}^m \to \mathbb{F}$

$$f_{\vec{x}_0}(\vec{x}) = \begin{cases} p(\vec{x}) & \vec{x} \neq \vec{x}_0 \\ p(\vec{x}) + 1 & \vec{x} = \vec{x}_0 \end{cases}$$

- $f_{\vec{x}_0}(\vec{x}) \notin \mathbb{F}_{\le d}[x_1, \ldots, x_m]$
- $\underset{\vec{x}_0, r}{P}[V^{f_{\vec{x}_0}, \pi} \underset{\text{on random } r}{\text{accepts}}] \ge \underset{\vec{x}_0, r}{P}[V^{p, \pi} \underset{\text{on random. } r}{\text{accepts}}] - \frac{q}{|\mathbb{F}^m|} \ge 1 - \frac{q}{|\mathbb{F}^m|}$

\# of queries performed by V

the proof associated with p

$\Rightarrow$ There exists $\vec{x}_0 \in \mathbb{F}^m$, such that

$$\exists \pi \ \underset{r}{P}[V^{f_{\vec{x}_0}, \pi} \underset{\text{on random } r}{\text{accepts}}] \ge 1 - \frac{q}{|\mathbb{F}^m|} > \frac{1}{2}$$

Hence, there is <u>no</u> tester V making $q < |\mathbb{F}^m|/2$ queries to func. and proof sat. the following:

<u>Comp</u> $p \in \mathbb{F}_{\le d}[x_1, \ldots, x_m] \Rightarrow \exists \pi \ \underset{r}{P}[V^{p, \pi} \underset{\text{on random } r}{\text{accepts}}] = 1$

<u>Sound</u> $f \notin \mathbb{F}_{\le d}[x_1, \ldots, x_m] \Rightarrow \forall \pi \ \underset{r}{P}[V^{f, \pi} \underset{\text{on random } r}{\text{accepts}}] \le \frac{1}{2}$

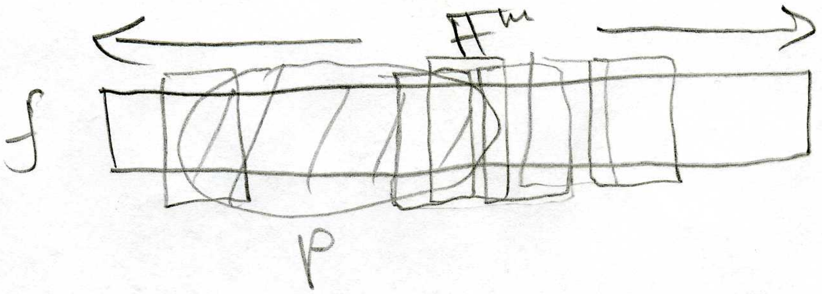
(2) LDT Theorem $\Rightarrow$ low degree tester

LDT Theorem For any $f: \mathbb{F}^m \to \mathbb{F}$,

$$\left| agr_{\le d}(f) - \mathop{\mathbb{E}}_{s \in S_2^m}\left[agr_{\le d}(f_{|s})\right]\right| \le m^{O(1)} \cdot \left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$$

$\max_{p \in \mathbb{F}_{\le d}[x_1 \ldots x_m]} \Pr_{\vec{x}}[f(\vec{x}) = p(\vec{x})]$

all planes in $\mathbb{F}^m$



Reminder Parameters:

$d = m \cdot (h-1)$
$= \text{polylog} n$

$h^m = n$

$h = \log n$
$m = \frac{\log n}{\log\log n}$

$|\mathbb{F}|$ chosen so that $m^{O(1)} \cdot \left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$ in the above theorem is small.

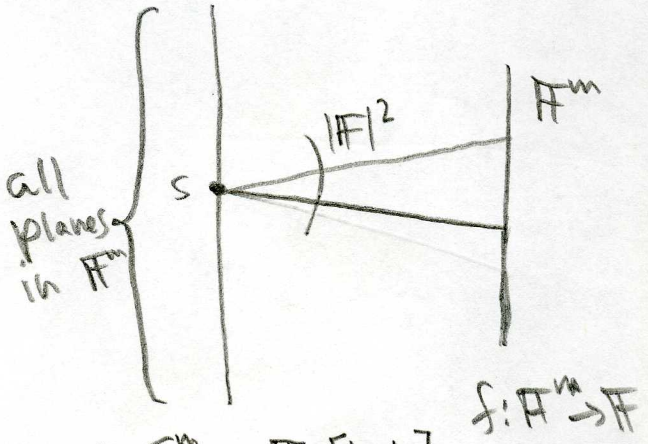
$|\mathbb{F}| = \text{polylog} n$.

Plane vs. Point tester

1. Pick u.a.r $s \in S_2^m$, $\vec{x} \in s$.
   "uniformly at random"

2. Check $\pi(s)(t_1,t_2) \stackrel{?}{=} f(\vec{x})$
   where $\vec{x} = s(t_1,t_2)$

canonical representation of s as a func.
$s: \mathbb{F}^2 \to \mathbb{F}^m$



$\pi: S_2^m \to \mathbb{F}_{\le d}[t_1,t_2]$

<u>Claim</u> <u>Assuming LDT Theorem</u>,

<u>Comp</u> $p \in \mathbb{F}_{\le d}[x_1,\dots,x_m] \Rightarrow \exists \pi \quad \underset{s,\vec{x}\in s}{P}\left(\text{Plane-vs-Point accepts given access to } f,\pi\right) = 1.$

<u>Sound</u> $agr(f, \mathbb{F}_{\le d}[x_1,\dots,x_m]) \le 0.8 \Rightarrow \forall \pi$

$$\underset{s,\vec{x}\in s}{P}\left(\begin{array}{l}\text{Plane-vs-Point accepts}\\ \text{given access to } f,\pi\end{array}\right) \le 0.9$$

<u>Note</u> LDT Thm gives an almost-tight result for <u>all</u> the spectrum.

<u>Pf</u> Comp follows for $\pi(s) \doteq f_{|s}$.

Sound: assuming $agr(f, \mathbb{F}_{\le d}[x_1,\dots,x_m]) \le 0.8$. Fix some π.

$$\underset{s,x\in s}{P}\left(\text{Plane-vs-Point}^{f,\pi}\right) \le \underset{s\in S_2^m}{\mathbb{E}} agr_{\le d}(f_{|s}) \le agr_{\le d}(f) + m^{O(1)}\left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$$

choosing closest to $f_{|s}$ is best possible π

≤ 0.9

for app. $|\mathbb{F}|$

□

Today Will show:

$$\mathrm{agr}_{\le d}(f) \ge \mathop{\mathbb{E}}_{s\in S_2^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right] - m^{O(1)}\cdot\left(\frac{d}{|\mathbb{F}|}\right)^{-\Omega(1)}$$

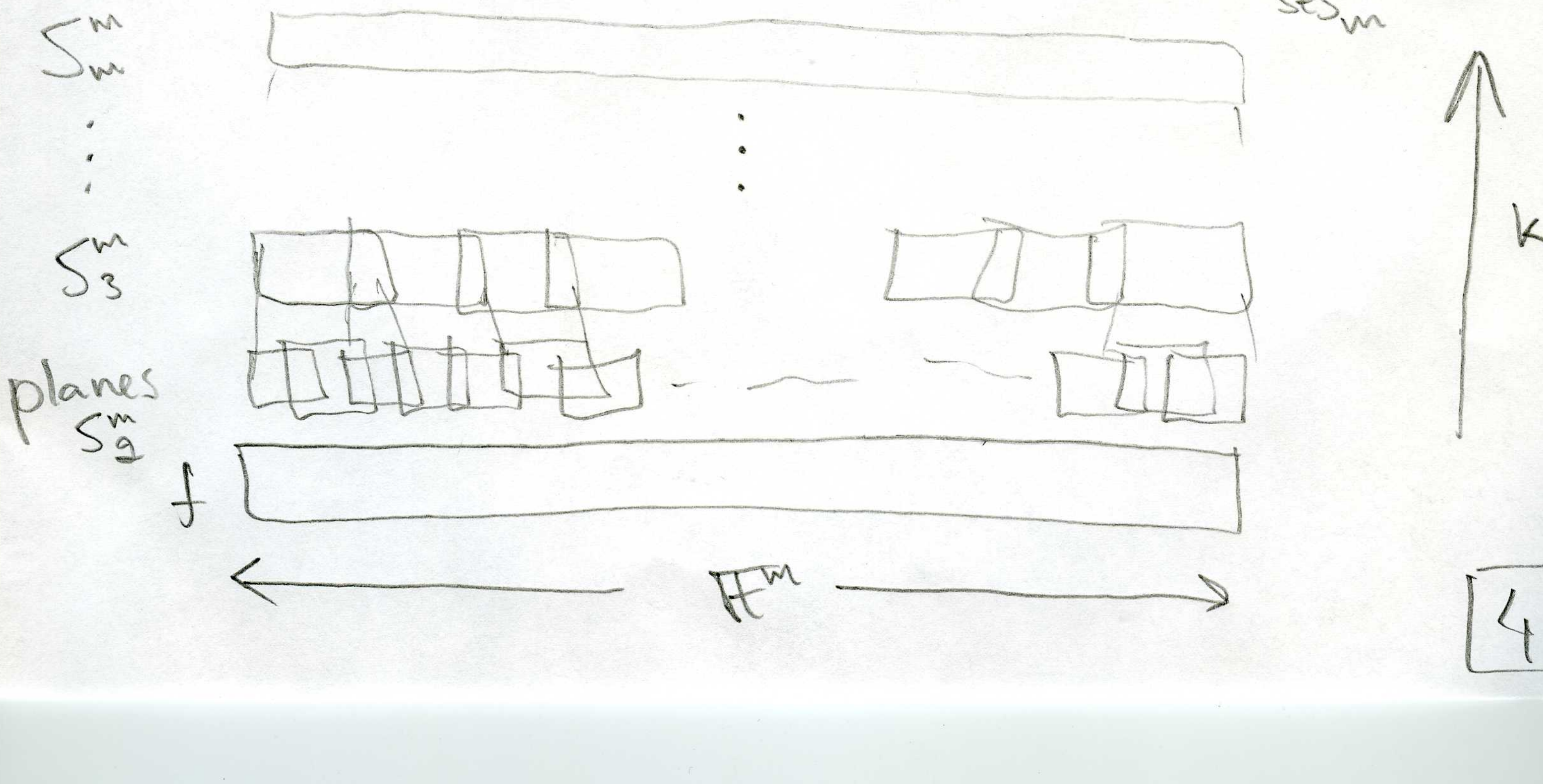
Together with Ex, this will conclude the proof of the LDT Theorem.

(and hence the proof that $NP \subseteq PCP[O(\log n), O(1)]_{\{0,1\}^{\mathrm{polylog}\,n}}$)

Proof strategy Will show by induction that for every $2 \le k \le m$,

$$\mathop{\mathbb{E}}_{s\in S_k^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right] \ge \mathop{\mathbb{E}}_{s\in S_2^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right] - m^{O(1)}\cdot\left(\frac{d}{|\mathbb{F}|}\right)^{-\Omega(1)}$$

In particular, for $k=m$. Note $\mathrm{agr}_{\le d}(f) = \mathop{\mathbb{E}}_{s\in S_m^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right]$



__Important Observation__ "Symmetry"

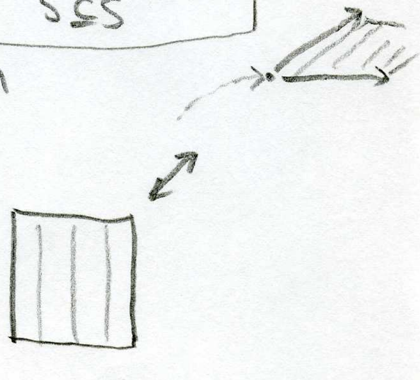
> __Note__ for every $s' \in S_k^m$, there's the same # of $s \in S_{k'}^m$, $k' > k$ s.t. $s' \subseteq s$

For every $s \in S_{k+1}^m$,

there's a bijection $s \longleftrightarrow \mathbb{F}^{k+1}$

(given by the canonical representation)

$s' \in S_k^m,\ s' \subseteq s \ \leftrightarrow\ s' \in S_k^{k+1}$



Hence, suffices to prove for $f : \mathbb{F}^{k+1} \to \mathbb{F}$,

$$(*)\quad agr_{\le d}(f) \ge \mathop{\mathbb{E}}_{s \in S_k^{k+1}}\left[agr_{\le d}(f_{|s}) \right] - \varepsilon \qquad \left[\text{where } \varepsilon = \left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)} \right]$$

Since

$$\mathop{\mathbb{E}}_{s \in S_k^m}\left[agr_{\le d}(f_{|s})\right] \overset{(*)}{\ge} \mathop{\mathbb{E}}_{s \in S_k^m}\left[\mathop{\mathbb{E}}_{s' \in S_k^{k+1}}\left[agr_{\le d}(f_{|s'})\right] - \varepsilon \right]$$

$$\overset{(*)}{\ge} \mathop{\mathbb{E}}_{s \in S_k^m}\left[\mathop{\mathbb{E}}_{s' \in S_k^{k+1}}\left[\mathop{\mathbb{E}}_{s'' \in S_{k-1}^{k}}\left[agr_{\le d}(f_{|s''})\right] - \varepsilon \right] - \varepsilon - \varepsilon \right]$$

$$\dots$$

$$\overset{(*)}{\ge} \mathop{\mathbb{E}}_{s \in S_2^m}\left[agr_{\le d}(f_{|s})\right] - \varepsilon m$$

Bootstrapping argument

We will only prove :

$$\mathrm{agr}_{\leq 2d}(f) \geq \mathop{\mathbb{E}}_{s \in S_k^{k+1}}\left[\mathrm{agr}_{\leq d}(f_{|s})\right]^2 - \varepsilon \qquad \varepsilon \leq O\left(\sqrt{\tfrac{d}{|\mathbb{F}|}}\right)$$

In ex.#3, Show this *implies*

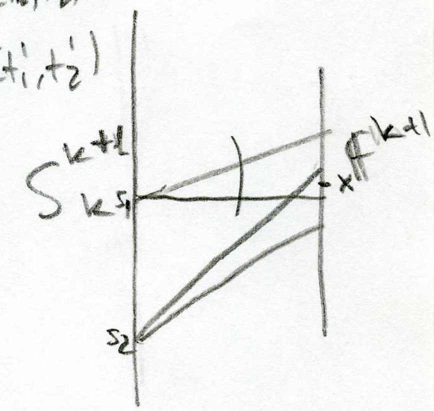
$$\mathrm{agr}_{\leq d}(f) \geq \mathop{\mathbb{E}}_{s \in S_k^{k+1}}\left[\mathrm{agr}_{\leq d}(f_{|s})\right] - \varepsilon' \qquad \varepsilon' \leq \left(\tfrac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$$

Hyperplanes Consistency

$\pi : S_k^{k+1} \to \mathbb{F}_{\leq d}[t_1, \ldots, t_k]$.

$\vec{x} = s_1(t_1, t_2)$

$\vec{x} = s_2(t_1', t_2')$



Claim For any $f : \mathbb{F}^{k+1} \to \mathbb{F}$,

$$\mathop{P}_{\substack{s_1, s_2 \in S_k^{k+1} \\ \vec{x} \in s_1, s_2}}\left(\pi(s_1)(t_1, t_2) = f(\vec{x}) = \pi(s_2)(t_1', t_2')\right) \geq$$

$$\mathop{P}_{s, \vec{x} \in s}\left(\pi(s)(t_1, t_2) = f(\vec{x})\right)^2$$

$s(t_1, t_2) = \vec{x}$

Note possibly $s_1 = s_2$; counting triplets $(s_1, s_2, \vec{x})$ s.t $\vec{x} \in s_1, s_2$

Technique Counting + Convexity

$$|\mathbb{F}^m| \cdot |\{s \in S_k^{k+1} \mid x \in s\}|^2$$

Some x

$\forall x$ this is the same

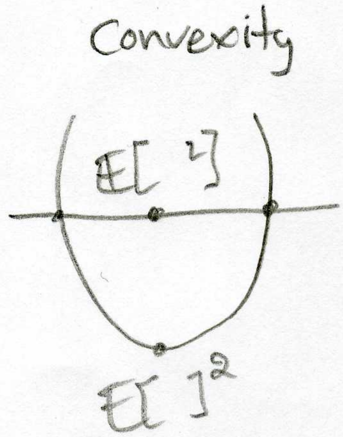
Proof Abbrev. $I_{s,x}$ = indicator for $\pi(s)(t_1,t_2) = f(\vec{x})$ where $\vec{x} = s(t_1,t_2)$

Want to show:

$$\mathop{\mathbb{E}}_{\substack{s_1,s_2\\ x\in s_1,s_2}}\left[I_{s_1,x}\cdot I_{s_2,x}\right] \geq \left(\mathop{\mathbb{E}}_{s,\,x\in s}\left[I_{s,x}\right]\right)^2$$

$$\parallel$$

$$\mathop{\mathbb{E}}_{x}\left[\left(\mathop{\mathbb{E}}_{s\ni x}\left[I_{s,x}\right]\right)^2\right]$$



Inequality follows from Jensen.

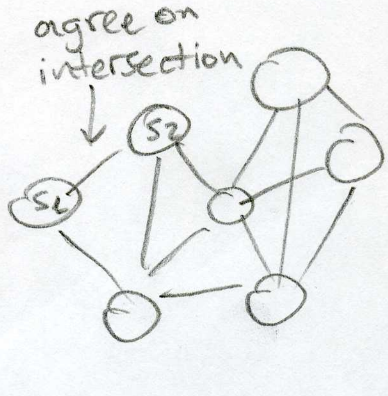
$\square$

Claim For every s_1, s_2 that intersect on a $(k-1)$-dim affine sub.,
$\mathop{\mathbb{E}}_{\vec{x}\in s_1,s_2}\left[I_{s_1,\vec{x}}\cdot I_{s_2,\vec{x}}\right] > \frac{d}{|\mathbb{F}|} \Longrightarrow \pi(s_1)$ & $\pi(s_2)$ agree on intersec.

Proof Schwartz-Zippel. $\square$

Corr $\mathop{P}_{\substack{s_1,s_2\\ x\in s_1,s_2}}\left[\pi(s_1)\,\&\,\pi(s_2) \text{ agree on intersection} + \pi(s_1)\,\&\,\pi(s_2) \text{ agree with } f \text{ on } \vec{x}\right] \geq \mathop{\mathbb{E}}_{s\in S_k^{k+1}}\left[agr_{\leq d}(f_{|s})\right]^2 - \frac{d}{|\mathbb{F}|}$

Hyperplanes Graph



Vertices = hyperplanes S_k^{k+1}

Edges = (s_1,s_2) s.t. s_1, s_2 agree on their intersection (possibly $s_1\cap s_2=\emptyset$).

Note I Graph is dense by Corr.

II This is not true for dimensions $< k$.

The LDT pf was shown in two subsequent classes. This is the reminder for the beginning of the second class.

<u>Recall</u> (where we are within the proof of the LDT Theorem)

<u>Want</u> $\quad agr_{\le d}(f) \ge \underset{S \in S_2^m}{\mathbb{E}}\left[agr_{\le d}(f_{|S})\right] - m\left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$

$f: \mathbb{F}^m \to \mathbb{F}$

<u>Reduced that to proving:</u> $\quad f: \mathbb{F}^{k+1} \to \mathbb{F}$

$$agr_{\le 2d}(f) \ge \left(\underset{S \in S_k^{k+1}}{\mathbb{E}}\left[agr_{\le d}(f_{|S})\right]\right)^2 - m\sqrt{\frac{d}{|\mathbb{F}|}}$$

<u>Defined hyperplanes graph</u>

Vertices = hyperplanes = S_k^{k+1}

Edges = (S_1, S_2) s.t. $\pi(S_1), \pi(S_2)$ agree on intersection.

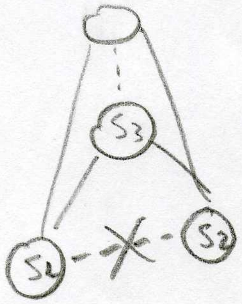
<u>Denote</u> $\quad \gamma \doteq \underset{S \in S_k^{k+1}}{\mathbb{E}}\left[agr_{\le d}(f_{|S})\right]$

<u>Showed</u> $\quad \underset{\substack{S_1, S_2 \in S_k^{k+1} \\ \vec{x} \in S_1, S_2}}{P}\left[(S_1, S_2) \in E \wedge \underset{\text{agree w/ } f \text{ on } \vec{x}}{\pi(S_1), \pi(S_2)}\right] \ge \gamma^2 - \left(\frac{d}{|\mathbb{F}|}\right)$

<u>Plan</u> continue analyzing the graph and use to conclude.

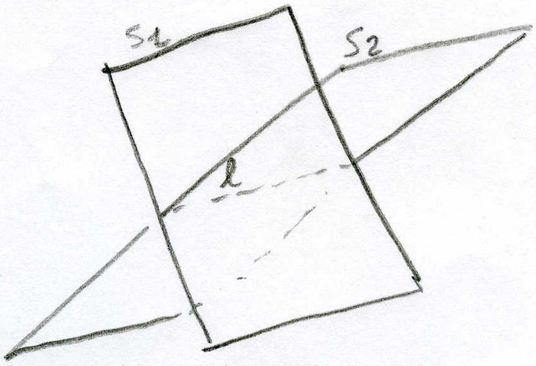
<u>Claim</u> ("Almost-Transitivity") For any $s_1, s_2 \in V$,

$$(s_1, s_2) \notin E \implies \underset{s_3 \in V}{P}\left((s_1, s_3) \in E \wedge (s_2, s_3) \in E \right) \leq \frac{d+1}{|\mathbb{F}|}$$



<u>Pf</u> $(s_1, s_2) \notin E \Rightarrow$ there exists l an affine subspace of dim $(k-1)$ s.t. $l \subseteq s_1, s_2$.

$\pi(s_1)_{|l} \not\equiv \pi(s_2)_{|l}$



<u>Note</u> This is not true for dimensions > k+1

"with probability"

Pick u.a.r $s_3 \in V$.

<u>Bad event #1</u> $s_3 \cap l = \emptyset$. Happens w.p. $\frac{1}{|\mathbb{F}|}$

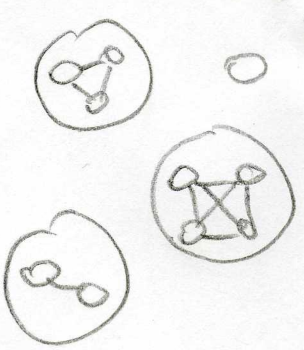
<u>Bad event #2</u> $s_3 \cap l \neq \emptyset$, but $\pi(s_1)$ & $\pi(s_2)$ agree on the intersection $s_3 \cap l$. Happens w.p $\leq \frac{d}{|\mathbb{F}|}$

If neither bad event happens, $\pi(s_3)$ does not agree either with $\pi(s_1)$ or with $\pi(s_2)$.

$\square$

<u>Remark</u> This argument works for planes and higher dim aff. sub., but not for lines.

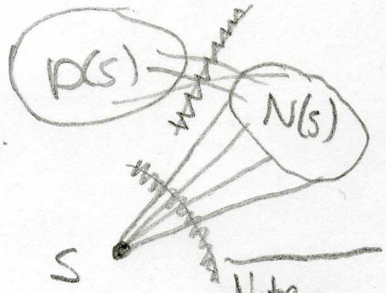
Claim: Can remove $\le 3\cdot\sqrt{\frac{d+1}{|\mathbb{F}|}}\,|V|^2$ edges and partition graph into disjoint cliques.
(transitive)



Pf: Partitioning algorithm:

While possible, choose $s \in V$ whose connected component is not a clique. Let $N(s)$ = neighbors of s.
$D(s)$ = members of cc which are not neighbors.

- If $|N(s)| \le \sqrt{\frac{d+1}{|\mathbb{F}|}}\,|V|$, remove edges $\{s\}\times N(s)$.



- Otherwise, remove edges $N(s)\times D(s)$.

Note: endpoints of removed edges are never in the same cc again

Note that when algo halts, graph is transitive.

How many edges removed?

- At most $|V|\cdot\sqrt{\frac{d+1}{|\mathbb{F}|}}\,|V| = \sqrt{\frac{d+1}{|\mathbb{F}|}}\,|V|^2$

Fix iteration s.

- For every $s' \in D(s)$, $(s,s') \notin E$

almost trans. $\Longrightarrow$ at most $\frac{d+1}{|\mathbb{F}|}|V|$ $s'' \in N(s)$ connected to s'.

$$\Rightarrow \frac{|N(s)\times D(s) \cup D(s)\times N(s)|}{|N(s)|\cdot|D(s)|} \le \frac{2\cdot\frac{d+1}{|\mathbb{F}|}|V|\cdot\cancel{|D(s)|}}{|N(s)|\cdot\cancel{|D(s)|}} \le 2\sqrt{\frac{d+1}{|\mathbb{F}|}}$$

$\square$

Claim If a disjoint union of cliques has $\gamma \cdot |V|^2$ edges, there is a clique of size $\geq \gamma|V|$

Pf Assume there are t cliques of sizes

$$c_1 \geq c_2 \geq \cdots \geq c_t$$

Then,

$$\gamma|V|^2 \leq \sum_{i=1}^{t} c_i^2 \leq \sum_{i=1}^{t} c_1 \cdot c_i = c_1 \sum_{i=1}^{t} c_i = c_1 \cdot |V|$$

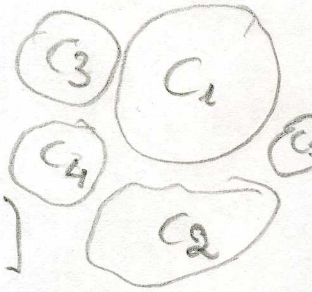
$\Rightarrow c_1 \geq \gamma|V|.$

Practice; redundant

$\square$

Weight For $e=(s_1,s_2)\in E$, let

$$\omega(e) \doteq \underset{x\in s_1\cap s_2}{P}\left[\pi(s_1) \,\&\, \pi(s_2) \text{ agree with } f \text{ on } \vec{x}\right]$$



For $s\in V$, let $\omega(s) \doteq \underset{\vec{x}\in s}{P}\left[\pi(s) \text{ agrees with } f \text{ on } \vec{x}\right]$

Claim The prev. claim holds when replacing "size" with "weight", i.e., $\underset{e\in E}{\mathbb{E}}[\omega(e)] \geq \gamma \Rightarrow \exists C \subseteq V \ \underset{s\in C}{\mathbb{E}}[\omega(s)] \geq \gamma$

Pf For a clique $C \subseteq V$, let $\omega(C) \doteq \underset{s\in C}{\mathbb{E}}[\omega(s)]$

Assume wlog, there are t cliques

$$C_1 \cdots C_t \subseteq V$$

Then there exists i with $\omega(C_i) \geq \gamma$, since

$$\gamma \leq \underset{e\in E}{\mathbb{E}}[\omega(e)] \leq \underset{i}{\mathbb{E}} \underset{e\in E(C_i)}{\mathbb{E}}[\omega(e)] \leq \underset{i}{\mathbb{E}} \underset{s\in C_i}{\mathbb{E}}[\omega(s)] = \underset{i}{\mathbb{E}}\, \omega(C_i)$$

$\square$

Heavy Clique

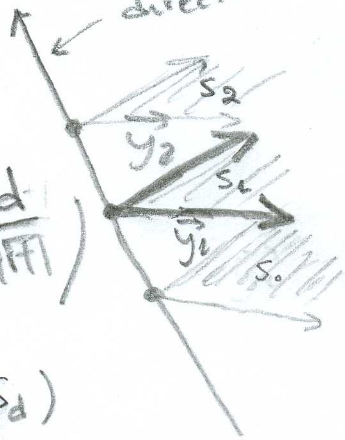
← the pf in this section is somewhat sketchy. The details should be worked out.

<u>Claim</u> A clique $C \subseteq V$ of weight $\gamma > \frac{2d+1}{|\mathbb{F}|}$ corr. to poly $p \in \mathbb{F}_{\leq 2d}[t_1 \dots t_k]$ s.t. $\underset{\vec{x} \in \mathbb{F}^{k+1}}{P}\left(p(\vec{x}) = f(\vec{x})\right) \geq \gamma$.

$\left[\Rightarrow agr_{\leq 2d}(f) \geq \gamma\right]$

<u>Pf</u> There are k lin. ind. directions $\vec{y}_1 \dots \vec{y}_k \in \mathbb{F}^{k+1}$ s.t. at least 2d+1 of their shifts $\overbrace{}^{(S_0 \dots, S_{2d})}$ are k-dim aff. sub. in C (because there are $|\mathbb{F}|$ possible shifts, and

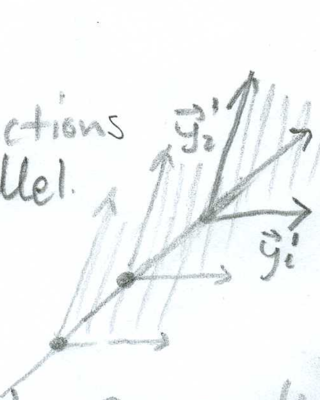
$$\underset{\vec{y}_0 \in \mathbb{F}^{k+1}}{\mathbb{E}} \ \underset{\vec{y}_1 \dots \vec{y}_k}{\mathbb{E}}\left[\text{sub. def by } \vec{y}_0, \vec{y}_1 \dots \vec{y}_k \in C\right] \geq \frac{2d}{|\mathbb{F}|}\Big)$$



Interpolate the polynomials $\pi(S_0) \dots \pi(S_d)$ to get p of $\deg \leq 2d$.

There necessarily exists another $\vec{y}_1' \dots \vec{y}_k' \in \mathbb{F}^{k+1}$ that are <u>not parallel</u> to $\vec{y}_1 \dots \vec{y}_k$ with the same property. → i.e., their ortho. directions are not parallel.

(Since prob. for parallel is $\leq \frac{1}{|\mathbb{F}|}$)



All 2d+1 new sub. must agree w/ p (they belong to same clique as prev sub.)

details omitted
$\Rightarrow$ all sub. in C must agree w/ p.

Summary

Heavy clique

$$\mathrm{agr}_{\le 2d}(f) \ge \max_{\text{clique } C} \omega(C) - O\left(\frac{d}{|\mathbb{F}|}\right)$$

disjoint union $\longrightarrow$ $\ge \mathbb{E}_{e \in E'}[\omega(e)] - O\left(\frac{d}{|\mathbb{F}|}\right)$

E' = edges after removal

partitioning algo $\longrightarrow$ $\ge \mathbb{E}_{e \in E}[\omega(e)] - O\left(\sqrt{\frac{d}{|\mathbb{F}|}}\right)$

density calculation (via convexity) $\longrightarrow$ $\ge \mathbb{E}_{S \in S_k^{k+1}}[\mathrm{agr}_{\le d}(f_{|S})]^2 - O\left(\sqrt{\frac{d}{|\mathbb{F}|}}\right)$

This concludes the proof of the LDT Thm.

$\square$