Probabilistically Checkable Proofs Spring 2008

Exercise no. 5

Due: June 18th, 2008

In this exercise you are asked to complete some details in the combinatorial proof of the PCP Theorem.

Throughout this exercise, l is a natural parameter and $G = (V, E)$ is a d-regular connected undirected multi-graph with self-loops (d, l are constants). We refer to the following random walk distribution $\mathcal{W}$ over $V \times V$:

- Pick a vertex $v_0 \in V$ uniformly at random.
- Perform a random walk on G starting v_0. After each step, stop with probability $\frac{1}{l}$.
- Let v_t be the ending vertex. Output the pair (v_0, v_t).

1. **Independence and conditioning.**

   (a) Let X and Y be random variables whose joint distribution is given by the matrix:

   $$\begin{pmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 & 0.1 & 0.1 \end{pmatrix}$$

   Are X and Y independent? If not, describe a random variable Z such that for any k, X and Y are independent conditioning on $Z = k$.

   (b) Show that the random variables v_0 and v_t defined by the random walk above are dependent.

   (c) Let $e = (u, v) \in E$ be an edge and let P_e be the random variable counting the number of steps $u \to v$ in the walk. Show that for any $k \geq 1$, the random variables v_0 and v_t are independent conditioned on $P_e = k$.

   (d) Fix $k \geq 1$. Show that the distribution of v_0 conditioned on $P_e = k$ is the distribution generated by a random walk that starts in u and stops before every step with probability $\frac{1}{l}$, and that the distribution of v_t conditioned on $P_e = k$ is the distribution generated by a random walk that starts in v and stops before every step with probability $\frac{1}{l}$.

2. **From distributional constraints to constraint graph.** For a natural number i, define the i'th power graph $G^i = (V, E^i)$ to be the multi-graph in which there is an edge $(u, v) \in E^i$ for every path (not necessarily simple) of length exactly i from u to v in G.

   Let W denote the matrix representing the joint distribution of v_0 and v_t in the random walk distribution $\mathcal{W}$.

   (a) **Powers.** Let A denote the adjacency matrix of G, i.e., for every $(u, v) \in V \times V$ let $A(u, v)$ be the number of edges between u and v in G. Show that the adjacency matrix of G^i is A^i.

(b) **Convex combination.** Let W^i be the normalized adjacency matrix $W^i = \frac{1}{|V|d^i} \cdot A^i$. Find a sequence of fractions $\{\alpha_i\}_{i=1}^{\infty}$, $\sum_{i=1}^{\infty} \alpha_i = 1$, such that $W = \sum_{i=1}^{\infty} \alpha_i W^i$.

(c) **Graph.** Let $\delta > 0$. Show that there exists a natural number $c = c(\delta)$ as follows. There exists a multi-graph $G^* = (V, E^*)$ with an adjacency matrix A^* and a normalized adjacency matrix W^*, such that: (i) $|E^*| \leq c \cdot |V|$; (ii) $|W^* - W| \leq \delta$, where we define $|W^* - W| = \frac{1}{2} \cdot \sum_{(u,v) \in V} |W^*(u, v) - W(u, v)|$.

(d) **Powering.** Complete the randomness-efficient soundness amplification (aka *powering*) transformation $\langle G = (V, E), \Sigma, \Phi \rangle \mapsto \langle G^* = (V^*, E^*), \Sigma^*, \Phi^* \rangle$ satisfying:

- *Completeness:* $\textsc{Sat}(G) = 1 \Rightarrow \textsc{Sat}(G^*) = 1$
- *Soundness:* $\textsc{UnSat}(G) \geq \varepsilon \Rightarrow \textsc{UnSat}(G^*) \geq \Omega(l \cdot \min\left\{\varepsilon, \frac{1}{l}\right\})$
- $|G^*| = O(|G|)$; $|\Sigma^*| = |\Sigma|^{d^l}$

Recall that in class we analyzed the randomness-efficient soundness amplification resulting in distributional constraints. You are asked to transform the distributional constraints into a constraint graph.

3. **Expanderization.** Show how to transform a constraint graph $\langle G = (V, E), \Sigma, \Phi \rangle$ where G is a d-regular undirected multi-graph into a constraint graph $\langle G^* = (V, E^*), \Sigma, \Phi^* \rangle$, where:

   - G^* is a $(|V|, d', \lambda)$-expander that has self-loops, where d' is a constant and $\lambda < d'$.
   - $|G^*| = O(|G|)$
   - $\Omega(\textsc{UnSat}(G)) \leq \textsc{UnSat}(G^*) \leq \textsc{UnSat}(G)$

4. **Tightness.** Show that for every natural parameter l, for infinitely many d's, there exists a constraint graph $\langle G = (V, E), \Sigma, \Phi \rangle$ such that for G^* that is obtained from G via the randomness-efficient soundness amplification with parameter l, it holds that $\textsc{UnSat}(G) \geq \frac{1}{2} - O(\frac{1}{\sqrt{d}})$, while $\textsc{UnSat}(G^*) \leq \frac{1}{2}$.

   - You may use the following claim without proof. For infinitely many natural numbers d, for infinitely many n's, there is a d-regular graph G on n vertices such that: (i) G has girth $\frac{2}{3} \log_d n$ (*girth* is the length of the shortest cycle in a graph); (ii) $\lambda(G) = 2\sqrt{d-1}$.
   - Why does not your proof contradict the correctness of the transformation stated in Question 2d?