The LDT pf was shown in two subsequent classes. This is the reminder for the beginning of the second class.

<u>Recall</u> (where we are within the proof of the LDT Theorem)

<u>Want</u> $\mathrm{agr}_{\le d}(f) \geq \mathop{\mathbb{E}}_{S \in S_2^m}\left[\mathrm{agr}_{\le d}(f_{|S})\right] - m\left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$

$f: \mathbb{F}^m \to \mathbb{F}$

<u>Reduced that to proving:</u> $f: \mathbb{F}^{k+1} \to \mathbb{F}$

$$\mathrm{agr}_{\le 2d}(f) \geq \left(\mathop{\mathbb{E}}_{S \in S_k^{k+1}}\left[\mathrm{agr}_{\le d}(f_{|S})\right]\right)^2 - m\sqrt{\frac{d}{|\mathbb{F}|}}$$

<u>Defined hyperplanes graph</u>

Vertices = hyperplanes = S_k^{k+1}

Edges = (S_1, S_2) s.t. $\Pi(S_1), \Pi(S_2)$ agree on intersection.

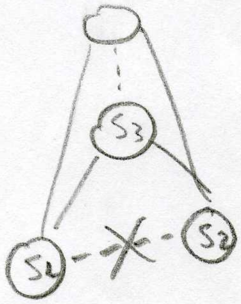
<u>Denote</u> $\gamma \doteq \mathop{\mathbb{E}}_{S \in S_k^{k+1}}\left[\mathrm{agr}_{\le d}(f_{|S})\right]$

<u>Showed</u> $\mathop{P}_{\substack{S_1, S_2 \in S_k^{k+1} \\ \vec{x} \in S_1, S_2}}\left[(S_1, S_2) \in E \wedge \underset{\text{agree w/ } f \text{ on } \vec{x}}{\Pi(S_1), \Pi(S_2)}\right] \geq \gamma^2 - \left(\frac{d}{|\mathbb{F}|}\right)$

<u>Plan</u> continue analyzing the graph and use to conclude.

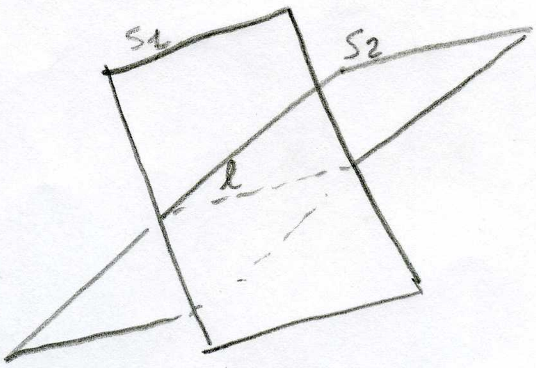
<u>Claim</u> ("Almost-Transitivity") For any $S_1, S_2 \in V$,

$$(S_1, S_2) \notin E \implies \underset{S_3 \in V}{P}\left((S_1,S_3) \in E \wedge (S_2,S_3) \in E \right) \leq \frac{d+1}{|\mathbb{F}|}$$



<u>Pf</u> $(S_1, S_2) \notin E \Rightarrow$ there exists l an affine subspace of dim $(k-1)$ s.t. $l \subseteq S_1, S_2$.

$\pi(S_1)_{|l} \not\equiv \pi(S_2)_{|l}$



<u>Note</u> This is not true for dimensions $> k+1$

"with probability"

Pick u.a.r $S_3 \in V$.

<u>Bad event #1</u> $S_3 \cap l = \emptyset$. Happens w.p. $\frac{1}{|\mathbb{F}|}$

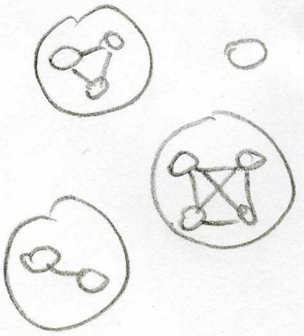
<u>Bad event #2</u> $S_3 \cap l \neq \emptyset$, but $\pi(S_1)$ & $\pi(S_2)$ agree on the intersection $S_3 \cap l$. Happens w.p $\leq \frac{d}{|\mathbb{F}|}$

If neither bad event happens, $\pi(S_3)$ does not agree either with $\pi(S_1)$ or with $\pi(S_2)$.

$\square$

<u>Remark</u> This argument works for planes and higher dim aff. sub., but not for lines.

<u>Claim</u> Can remove $\leq 3 \cdot \sqrt{\frac{d+1}{|\mathbb{F}|}} |V|^2$ edges and partition graph into disjoint cliques.
(transitive)

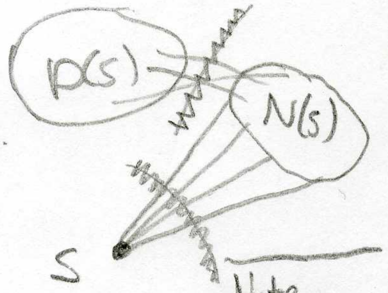


<u>Pf</u> Partitioning algorithm:

While possible, choose $s \in V$ whose connected component is not a clique. Let $N(s)$ = neighbors of s.
$D(s)$ = members of cc which are not neighbors.

- If $|N(s)| \leq \sqrt{\frac{d+1}{|\mathbb{F}|}} |V|$, remove edges $\{s\} \times N(s)$.

- Otherwise, remove edges $N(s) \times D(s)$.



<u>Note</u>
endpoints of removed edges are never in the same cc again

Note that when algo halts, graph is transitive.
How many edges removed?

- At most $|V| \cdot \sqrt{\frac{d+1}{|\mathbb{F}|}} |V| = \sqrt{\frac{d+1}{|\mathbb{F}|}} |V|^2$

Fix iteration s.
- For every $s' \in D(s)$, $(s, s') \notin E$

$\xRightarrow{\text{almost trans.}}$ at most $\frac{d+1}{|\mathbb{F}|} |V|$ $s'' \in N(s)$ connected to s'.

$$\Rightarrow \frac{|N(s) \times D(s) \cup D(s) \times N(s)|}{|N(s)| \cdot |D(s)|} \leq \cancel{\frac{2 \cdot \frac{d+1}{|\mathbb{F}|} |V| \cdot |D(s)|}{|N(s)| \cdot |D(s)|}} \leq 2 \sqrt{\frac{d+1}{|\mathbb{F}|}}$$

$\square$

Claim If a disjoint union of cliques has $\gamma \cdot |V|^2$ edges, there is a clique of size $\geq \gamma |V|$

Pf Assume there are t cliques of sizes

$$c_1 \geq c_2 \geq \cdots \geq c_t$$

Then,

$$\gamma |V|^2 \leq \sum_{i=1}^{t} c_i^2 \leq \sum_{i=1}^{t} c_1 \cdot c_i = c_1 \sum_{i=1}^{t} c_i = c_1 \cdot |V|$$

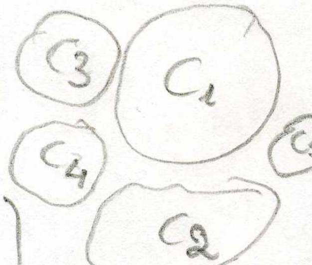
$\Rightarrow c_1 \geq \gamma |V|$.

Practice; redundant

$\square$

Weight For $e=(s_1, s_2) \in E$, let

$$w(e) \doteq \underset{x \in s_1 \cap s_2}{P} \left[\pi(s_1) \,\&\, \pi(s_2) \text{ agree with } f \text{ on } \vec{x} \right]$$



For $s \in V$, let $w(s) \doteq \underset{\vec{x} \in s}{P} \left[\pi(s) \text{ agrees with } f \text{ on } \vec{x} \right]$

Claim The prev. claim holds when replacing "size" with "weight", i.e., $\underset{e \in E}{\mathbb{E}}[w(e)] \geq \gamma \Rightarrow \exists C \subseteq V \;\; \underset{s \in C}{\mathbb{E}}[w(s)] \geq \gamma$

Pf For a clique $C \subseteq V$, let $w(C) \doteq \underset{s \in C}{\mathbb{E}} \left[w(s) \right]$

Assume wlog. there are t cliques

$$C_1 \cdots C_t \subseteq V$$

Then there exists i with $w(C_i) \geq \gamma$, since

$$\gamma \leq \underset{e \in E}{\mathbb{E}} [w(e)] \leq \underset{i}{\mathbb{E}} \; \underset{e \in E(C_i)}{\mathbb{E}} \left[w(e) \right] \leq \underset{i}{\mathbb{E}} \; \underset{s \in C_i}{\mathbb{E}} \left[w(s) \right] = \underset{i}{\mathbb{E}} \, w(C_i)$$

$\square$

Heavy Clique

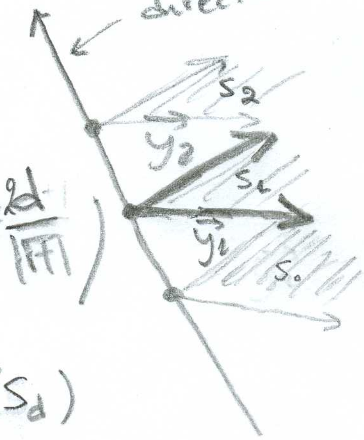
← the pf in this section is somewhat sketchy. The details should be worked out.

Claim A clique $C \subseteq V$ of weight $\gamma^{7\,\frac{2d+1}{|\mathbb{F}|}}$ corr. to poly $p \in \mathbb{F}_{\le 2d}[t_1 \dots t_k]$ s.t. $\underset{\vec{x} \in \mathbb{F}^{k+1}}{P}\left(p(\vec{x}) = f(\vec{x})\right) \ge \gamma$.

$\left[\Rightarrow agr_{\le 2d}(f) \ge \gamma\right]$

Pf There are k lin. ind. directions $\vec{y}_1 \dots \vec{y}_k \in \mathbb{F}^{k+1}$ s.t. at least 2d+1 of their shifts $(S_0 \dots, S_{2d})$ are k-dim aff. sub. in C (because there are $|\mathbb{F}|$ possible shifts, and

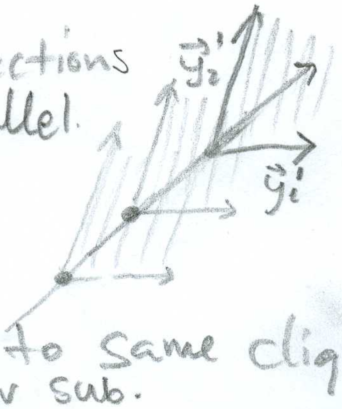
$$\underset{\vec{y}_0 \in \mathbb{F}^{k+1}}{\mathbb{E}} \; \underset{\vec{y}_1 \dots \vec{y}_k}{\mathbb{E}}\left[\text{sub. def by } \vec{y}_0, \vec{y}_1 \dots \vec{y}_k \in C\right] \ge \frac{2d}{|\mathbb{F}|}\Big)$$



Interpolate the polynomials $\pi(S_0) \dots \pi(S_d)$ to get p of deg $\le 2d$.

There necessarily exists another $\vec{y}_1\,' \dots \vec{y}_k\,' \in \mathbb{F}^{k+1}$ that are not parallel to $\vec{y}_1 \dots \vec{y}_k$ with the same property. → i.e., their ortho. directions are not parallel.

(Since prob. for parallel is $\le \frac{1}{|\mathbb{F}|}$)



All 2d+1 new sub. must agree w/ p (they belong to same clique as prev sub.)

details omitted
$\Rightarrow$ all sub. in C must agree w/ p.

Summary

Heavy clique

$$\mathrm{agr}_{\leq 2d}(f) \geq \max_{\text{clique } C} \omega(C) - O\left(\frac{d}{|\mathbb{F}|}\right)$$

disjoint union $\longrightarrow$ $$\geq \mathop{\mathbb{E}}_{e \in E'} [\omega(e)] - O\left(\frac{d}{|\mathbb{F}|}\right)$$

E' = edges after removal

partitioning algo $\longrightarrow$ $$\geq \mathop{\mathbb{E}}_{e \in E} [\omega(e)] - O\left(\sqrt{\frac{d}{|\mathbb{F}|}}\right)$$

density calculation (via convexity) $\longrightarrow$ $$\geq \mathop{\mathbb{E}}_{S \in S_k^{k+1}} \left[\mathrm{agr}_{\leq d}(f_{|S})\right]^2 - O\left(\sqrt{\frac{d}{|\mathbb{F}|}}\right)$$

This concludes the proof of the LDT Thm.

$\square$