CS388C: Combinatorics and Graph Theory — Review Sheet
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These are some of the concepts we assume in the class. If you have never learned them or need to review them, please take the time to do so. Some of these definitions come from *Combinatorial Problems and Exercises* by L. Lovasz.

1 Probability

Def: *conditional probability*

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

where A, B are two events.

Def: *independent events*

Two events, A and B are independent iff:

$$\Pr[A \cap B] = \Pr[A] \Pr[B]$$

Note: Equivalently: $\Pr[A|B] = \Pr[A]$ if $\Pr[B] \neq 0$ and $\Pr[B|A] = \Pr[B]$ if $\Pr[A] \neq 0$.

Def: *expectation*

Let X be a random variable over a set Λ with probability distribution $\Pr[x]$. Expectation of a function g(X) is

$$E[g(X)] = \sum_{x \in \Lambda} g(x) \Pr[x]$$

Expectation may also be taken over joint probability distributions:

$$E[f(X,Y)] = \sum_{(x,y) \in \Lambda} f(x,y) \Pr[x,y]$$

Think of: The pair (X, Y) may be thought of as a single random variable.

Note: Letting g be the identity, we get the definition of expected value.

Note: Expectation is a linear function. i.e.: $E[aX + bY + c] = aE[X] + bE[Y] + c$ for constants a, b, c and random variables X, Y. Try verifying this using the above definition.

2 Useful Inequalities

For real numbers:

- For all x, we have $1 + x \leq e^x$, with equality if and only if $x = 0$. For small x, we have $e^x \approx 1 + x$.
- For $k > 1$ and $x \geq -1$, we have $(1 + x)^k \geq 1 + kx$, with equality if and only if $x = 0$. For small x, we have $(1 + x)^k \approx 1 + kx$.

For factorials and binomial coefficients:

- Stirling's approximation for factorials:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

  In particular, we have the inequalities

$$\left(\frac{n}{e}\right)^n \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n} \left(\frac{n}{e}\right)^n.$$

- Using the weakest lower bound from Stirling, we can upper bound binomial coefficients:

$$\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k.$$

3 Set Theory

Def: *k-set*

A set of size k.

Def: $[n]$

$\{1, 2, \ldots, n\}$.

Def: $\binom{[n]}{k}$

Set of k-subsets of $[n]$.

Def: *Hamming distance*

$$d(a, b) = |\{i : a_i \neq b_i\}|$$

where a, b are two strings of the same length.

Think of: The number of different characters between the two strings.

Def: *incidence vector*

The incidence vector of a set $S \subseteq U$ (universe U) is a vector v whose entries are labeled with the members of U. $v_u = 1 \Leftrightarrow u \in S$ otherwise $v_u = 0$

Think of: The idea is similar to the adjacency matrix idea. Just put a 1 in all the columns labeled with elements of S

Def: *symmetric set difference*

$$S \triangle T = (S \cup T) - (S \cap T)$$

Think of: The elements that are in A or B, but not both. Kind of like set XOR.

Def: *maximal set*

A set S with some property such that no set strictly containing S has the property.

Note: The difference between maximal and maximum.

Def: *maximum set*

A set with largest possible size that has some property.

Note: The difference between maximal and maximum.

Def: *functions, vectors and strings*

$$\begin{aligned} f : A \to B &\quad \text{a function} \\ f_a = f(a) &\quad \text{a vector} \\ f(a_1), f(a_2), \ldots, f(a_{|A|}) &\quad \text{a string} \end{aligned}$$

Think of: $f : A \to B$ can be thought of as a vector that is indexed by elements of A with entries that are elements of B. You can pick some order to the elements of A to make that vector into a string.

Def: *poset* or *partially ordered set*

A set P with a relation $\preceq$ s.t.

- $\forall a, a \preceq a$
- $a \preceq b, b \preceq c \Rightarrow a \preceq c$
- $a \preceq b, b \preceq a \Rightarrow a = b$

4 Graph Theory

Def: *graph*

$G = (V, E)$ A set of vertices V and a set of edges E. Graph usually means undirected graph, where each edge is a subset of V of size 2.

Common Notation: $|V| = n$, $|E| = m$. i.e.: n vertices, m edges.

Def: *subgraph*

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. G' is a spanning subgraph if $V' = V$.

Def: *degree*

The number of edges incident to a vertex.

Def: *d-regular graph*

All vertices in the graph have degree d.

Def: *digraph*

A directed graph. Each edge is an ordered pair, i.e. $(a, b) \neq (b, a)$.

Def: *bipartite graph*

V can be separated into two sets A and B s.t. $A \cap B = \emptyset$, $A \cup B = V$ and $\{a, b\} \in E \Rightarrow a \in A, b \in B \vee a \in B, b \in A$.

Think of: A and B are each on a different side of a river and all of the edges in E are bridges over the river.

Def: *connected graph*

$\forall x, y \in V, \exists$ a path between x and y.

Def: *adjacency matrix*

A square matrix A where the rows and columns are labeled with the vertices of a graph G. $a_{i,j} = 1 \Leftrightarrow (i, j) \in E$ else $a_{i,j} = 0$.

Think of: Say we are looking at the row for vertex a. If there is an edge between a and b, put a 1 in b's column.

Note: For a bipartite graph, this matrix may have the set A as the rows and the set B as the columns for a more compact representation.

Def: *complement graph*

The complement of a graph G is a graph $\bar{G}$ with the same vertices as G, but $E(\bar{G}) = \{\{a, b\} : a \neq b, \{a, b\} \notin E(G)\}$.

Think of: Change empty spots in G to edges, change edges in G to empty spots.

Def: *distance*

The length of the shortest path between some $x, y \in V$.

Common Notation: $d(x, y)$.

Def: *diameter*

$\max_{x,y \in V} d(x, y)$.

Think of: The largest distance between two vertices.

Def: *cycle*

A sequence of vertices $v_1, \ldots, v_k$ s.t. $\{v_i, v_{i+1}\} \in E$ for $i = 1, \ldots, k-1$ and $\{v_k, v_1\} \in E$.

Think of: A "circle" of vertices.

Def: *girth*

The length of the smallest cycle in the graph.

Def: *tree*

A connected graph without cycles.

Note: Equivalently: A connected graph s.t. removing any edge disconnects it.

Note: Equivalently: A cycle-free graph s.t. introducing a new edge makes a cycle.

Def: *clique*

$K \subseteq V$ s.t. $\forall a, b \in K \Rightarrow \{a, b\} \in E$.

Think of: All pairs of vertices in K have an edge between them.

Common Notation: A *clique* of size n is K_n.

Common Notation: The size of the largest clique in a graph G is $\omega(G)$.

Def: *complete graph*

A graph on n vertices that is also a clique of n vertices.

Def: *independent set*

$I \subseteq V$ s.t. $\forall a, b \in I \Rightarrow \{a, b\} \notin E$.

Think of: The opposite of *clique*. Each pair of vertices in I must not have an edge between them.

Common Notation: The size of the largest independent set in a graph G is $\alpha(G)$.

Def: *vertex cover*

$A \subseteq V$ s.t. $\{a, b\} \in E \Rightarrow a \in A$ or $b \in A$.

Think of: Pick a set of vertices such that each edge has an end in the set.

Common Notation: $\tau(G)$ is the minimum size vertex cover.

Def: *edge cover*

$A \subseteq E$ s.t. $a \in V \Rightarrow \exists b, \{a, b\} \in A$.

Think of: Pick a set of edges such that each vertex is an end of at least one edge. This is somewhat similar to vertex cover, as the names suggest.

Def: *matching*

$A \subseteq E$ s.t. $\{a, b\}, \{c, b\} \in A \Rightarrow a = c$.

Think of: A set of edges so that no two elements in the set share an endpoint.

Common Notation: $\nu(G)$ is the maximum size matching.

Def: *perfect matching*

A matching of size $n/2$.

Think of: All vertices are matched.

Def: *chromatic number*

$c : V \to [r]$ is a coloring on the vertices. The chromatic number of a graph is the minimum r s.t. $(a, b) \in E \Rightarrow c(a) \neq c(b)$.

Common Notation: $\chi(G)$. Also, $[r] = \{1, 2, \ldots, r\}$.

Think of: Color the vertices such that the ends of each edge are different.

5 Abstract Algebra

Def: *field*

A set $\mathbb{F}$ and operations $+, *$ s.t. $\forall a, b, c \in \mathbb{F}$

- $a + b \in \mathbb{F}$, $ab \in \mathbb{F}$
- $a + b = b + a$, $ab = ba$
- $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$
- $a(b + c) = ab + ac$
- $\exists 0 \in \mathbb{F}$ s.t. $\forall a, a + 0 = a$
- $\exists 1 \in \mathbb{F}$ s.t. $\forall a, 1 * a = a$
- $\forall a, \exists -a \in \mathbb{F}$ s.t. $a + -a = 0$
- $\forall a \neq 0, \exists a^{-1} \in \mathbb{F}$ s.t. $a * a^{-1} = 1$

Think of: A set and operations that behave like the real numbers. Pretty much all theorems that hold for real numbers hold in fields, a notable exception being theorems involving limits.

Note: Finite fields may only have sizes that are powers of primes, i.e., size p^k for some prime p and positive integer k. A finite field with q elements is denoted $\mathbb{F}_q$.

Note: The integers *mod* a prime p are a finite field, denoted as $\mathbb{Z}_p = \mathbb{F}_p$.

Def: *monomial*

A product of non-negative powers of variables.

Example: x^2y^3z

Def: *polynomial*

Note: We will usually consider polynomials over finite fields, although the notes below apply to univariate polynomials over any field.

Note: For any $k+1$ distinct points, there is a unique ($\exists!$) polynomial of degree $\leq k$ passing through them.

Note: For a polynomial p, $p(a) = 0$ iff $(x-a)|p$, i.e., $p(x) = (x-a)q(x)$ for a polynomial q.

Note: A polynomial is monic if it has leading coefficient 1.

6 Linear Algebra

Def: *linearly independent*

A set of vectors $v_1, \ldots, v_k$ are L.I. if the only solution to

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_k v_k = 0$$

is $\lambda_i = 0, \forall i$.

Def: *eigenvalue*

λ is an eigenvalue of a matrix M if:

$$Mx = \lambda x$$

for some $x \neq 0$.

Note: The above x is called an *eigenvector*.

Def: *Vandermonde matrix*

A square matrix of the form

$$V = \begin{bmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \ldots & a_3^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{bmatrix}$$

Note: Vandermonde matrices are very useful when trying to find a polynomial going through a set of points.

Note: If the a_i's are distinct, V is invertible.

Def: *determinant*

You should be familiar with the cofactor expansion method of calculating determinants. But, you should also be aware of the "all possible permutations" definition:

$$\begin{vmatrix} M_{11} & M_{12} & \ldots & M_{1n} \\ M_{21} & M_{22} & \ldots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \ldots & M_{nn} \end{vmatrix} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) M_{1\pi_1} M_{2\pi_2} \cdots M_{n\pi_n}$$

where S_n is the group of all possible permutations of n elements.

Think of: One of the permutations in the sum is the identity permutation, where $\pi(i) = i$.

Def: *sign of a permutation* or $sgn(\pi)$

$$sgn(\pi) = (-1)^k$$

where k is the number of inversions in the permutation. An inversion is a pair x, y s.t. $x < y$ but $\pi(x) > \pi(y)$.