 

# Midterm

**Instructions.** There are 5 problems, each worth 10 points. Explain all answers. You may assume any theorems stated in class or in the book, unless the question is to prove such a theorem. You are allowed one sheet of paper, with writing on both sides. You have 75 minutes.

**Name:** ________________________________

1. Give a randomized algorithm for MAX-CUT, where the expected value of the cut for a graph on $n$ vertices and $m$ edges is $(1/2 + \Omega(1/n))m$. What constant in front of $1/n$ can you achieve? (Hint: you may assume that $n$ is even, and consider balanced cuts.)

2. Consider the following primality test:

   (a) If $n$ is even and greater than 2 or a perfect power then output COMPOSITE.

   (b) Choose $a_1, \dots, a_t$ uniformly at random from $\mathbb{Z}_n \setminus \{0\}$.

   (c) If for any $i$, $\gcd(a_i, n) \neq 1$ then output COMPOSITE.

   (d) Compute $b_i \leftarrow a_i^{(n-1)/2} \mod n$.

   (e) If for any $i$, $b_i \not\equiv \{\pm 1\} \mod n$, then output COMPOSITE.

   (f) If for all $i$, $b_i \equiv 1 \mod n$, then output COMPOSITE, else output PRIME.

   Unlike the algorithm we covered in class, this algorithm has 2-sided error. Show that if $n$ is prime, then the probability that the algorithm outputs COMPOSITE is small. How small is it?

3. Show that for a suitable constant $c$, after $cn \log_2 n$ balls are thrown into $n$ bins, with probability $1 - 1/n$ all bins have at least $\log_2 n$ balls.

4. A 3-coloring of an undirected graph $G = (V, E)$ is a function $c : V \to \{1, 2, 3\}$. An edge $\{u, v\}$ is properly colored if $c(u) \neq c(v)$. Our goal is to find a 3-coloring which properly colors as many edges as possible. Consider the algorithm which assigns a uniformly random coloring to the vertices. Assume that $m$, the number of edges in $G$, is at least $n$, the number of vertices.

   a) (2 points) For any edge $e$, let $A_e$ denote the event that $e$ is properly colored. Show that the events $A_e$ are not necessarily independent.

   b) (8 points) Show that with high probability, at least 65% of the edges are properly colored. To get full credit, you must show an error of $O(1/n)$ or, in the case when the maximum degree is $d$, an error $\exp(-\Omega(n/d^2))$. You get extra credit if you show both.

5. A cycle cover in a directed graph $G$ is a set of vertex-disjoint cycles which contain every vertex. Thus, if $G$ has $n$ vertices, then any cycle cover has $n$ edges. Give an efficient randomized algorithm to test whether a directed graph has a cycle cover, and show correctness.