CS395T: Randomized Algorithms | Problem Set No. 4
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1. Exercise 4.2 (page 75).

2. Problem 4.9.

3. Problem 4.14.

4. Let $\{X_i\}$ be a martingale with $X_0 = 0$ such that for each k, $|X_k - X_{k-1}| \le c$. Show that for any $t, \lambda > 0$,
$$\Pr[\max_{0\le i\le t} |X_i| \ge \lambda c\sqrt{t}] \le 2e^{-\lambda^2/2}.$$

5. The following balls and bins process models a simple distributed system where agents (balls) contend for resources (bins) but back off in the face of contention. The process evolves in rounds. In each round, balls are thrown independently and uniformly at random into n bins. Each ball that lands in a bin by itself is removed from the process. The remaining balls are thrown again in the next round. The process begins with $n/2$ balls and ends when all balls are removed.

   Show that with probability $1 - o(1)$, the process takes $O(\log\log n)$ rounds. (Hint: show that when there are b balls at the start of a round, with high probability there are not many more than b^2/n balls at the end of the round.)

   For extra credit, show that this process takes $O(\log\log n)$ rounds even when started with cn balls, for any constant $c > 0$.