CS395T: Pseudorandomness and Combinatorial Constructions
A Lower Bound on the Linear Expansion of Regular Graphs, 26 April 2001
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In this lecture, we shall discuss a result of Kahale from the paper *On the Second Eigenvalue and Linear Expansion of Regular Graphs*. For references and additional details on the proof, the reader may refer to the original work.

1 Introduction

An *expander graph* has the property that any small set of the vertices has a large number of neighbors. These graphs are useful in many different types of applications. For example, when designing a network, it may be desirable that every small collection of nodes be connected to a large number of other nodes. In this case, an expander graph would provide an excellent network topology. Expanders are also important for cryptography and for randomized algorithms. See lectures 11 to 13 for more details.

The *vertex expansion* or *expansion coefficient* of a subset of a graph is defined as the ratio between the number of unique neighbors of that set and the size of the set itself. We wish to find a lower bound on the vertex expansion of any small subset of a given expander graph. This bound essentially determines the quality of the expander. When attempting to prove properties of an algorithm which involves an expander, this type of result is critical because it guarantees a certain minimal expansion across all small subsets of the graph.

To obtain a lower bound, Tanner introduced the connection between the spectrum of a graph and its expansion. The basic result is that small second eigenvalues lead to large expansions. Tanner originally proved that for a subset X of a k-regular graph $G = (V, E)$,

$$\frac{|\partial X|}{|X|} \geq \frac{k^2}{\lambda^2 + (k^2 - \lambda^2)\frac{|X|}{|V|}},$$

where ∂X are the neighbors of the vertices in X and λ is the second largest eigenvalue of G in absolute value. It is also known that for an infinite family $\{G_n\}$ of k-regular graphs, the second eigenvalues are bounded below in the limit:

$$\liminf_{n\to\infty} |\lambda(G_n)| \geq 2\sqrt{k-1}.$$

A graph for which the second eigenvalue is no more than $2\sqrt{k-1}$ is called a *Ramanujan graph*. For Ramanujan graphs, Tanner's result yields the expansion coefficient

$$\frac{|\partial X|}{|X|} = \frac{k}{4} + o(k),$$

where X is a small set of vertices.

It turns out that we can obtain an even better estimate using second eigenvalue techniques. In this article, we improve the lower bound on the expansion coefficient to

$$\frac{|\partial X|}{|X|} \geq \frac{k}{2}\left[1 - \sqrt{\max\left\{0, 1 - \frac{4(k-1)}{\lambda^2}\right\}}\right]$$

as $|X|/|V| \to 0$. Therefore, a Ramanujan graph satisfies

$$\frac{|\partial X|}{|X|} \geq \frac{k}{2}$$

as $|X|/|V| \to 0$.

Kahale proves that this result is optimal by constructing a sequence of k-regular graphs $\{G_n\}$ which contain a small subset with expansion coefficient $k/2$ and for which the second eigenvalue is asymptotically ideal: $\lambda(G_n) \leq 2\sqrt{k-1}(1+o(1))$. This fact indicates that no better bound is possible using second eigenvalue techniques.

2 Notation

We shall denote a graph by $G = (V, E)$ where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Sometimes, we shall use the symbol n to indicate the size of the vertex set. We shall write $A = A(G)$ for the $n \times n$ adjacency matrix of a graph, whose entries satisfy $A_{ij} = 1$ whenever $(v_i, v_j) \in E$ and $A_{ij} = 0$ otherwise. For any matrix M with real eigenvalues, we shall denote the $(i+1)$-th largest eigenvalue by $\lambda_i(M)$ for a nonnegative index i. We shall frequently abbreviate $\lambda_i(A(G))$ by $\lambda_i(G)$.

For any two sets of vertices $X, Y \subseteq G$, we shall define the edge-counting function

$$e_G(X, Y) = |X \times Y \cap E|$$

which returns the number of edges of G which initiate in X and terminate in Y. This function depends critically on the topology of the graph. We may omit the subscript if it is clear which graph we are working with.

Let $\ell^2(V)$ be the set of real-valued functions on the set V. We may identify these functions as n-dimensional vectors. As usual, we compute the inner product of $\ell^2(V)$ functions by $f \cdot g = \sum_V f(v)\, g(v)$. Moreover, we have the following relationship: $(Af)(v) = \sum_{(v,w)\in E} f(w)$. Using this fact, we obtain

$$(Af) \cdot g = f \cdot (Ag) = \sum_{(v,w)\in E} f(v)\, g(w).$$

In particular, A is self-adjoint with respect to the $\ell^2(V)$ product. Finally, for any set of vertices $W \subseteq V$, the characteristic function of W is given by

$$\chi_W(v) = \begin{cases} 1 & \text{if } v \in W \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We also make several nonstandard definitions which figure prominently in the proof. For a sequence $Z_0, Z_1, \ldots, Z_t$ of nonempty subsets of V, let $\mathscr{Z} = \{Z_0, Z_1, \ldots, Z_t\}$. We shall denote the set of real-valued functions on $\mathscr{Z}$ using the symbol $\mathbb{R}^{\mathscr{Z}}$. Note that $\mathbb{R}^{\mathscr{Z}}$ is a $(t+1)$-dimensional vector space with exactly one component corresponding to each of the sets Z_i, and the canonical basis of $\mathbb{R}^{\mathscr{Z}}$ is the collection $\{\chi_{\{Z_0\}}, \chi_{\{Z_1\}}, \ldots, \chi_{\{Z_t\}}\}$. This unusual notation will make parts of the proof more transparent.

We also define a linear map on $\mathbb{R}^{\mathscr{Z}}$, $\Phi_G(\mathscr{Z})$ by the matrix

$$\Phi_{ij} = \frac{e(Z_i, Z_j)}{|Z_i|}.$$

This matrix may be interpreted as the adjacency matrix of a weighted, directed graph on $\mathbb{R}^{\mathscr{Z}}$ where the weight of edge (Z_i, Z_j) is the average number of edges from Z_i to Z_j per node of Z_i. This map will enable us to consider the average behavior of the graph over a set of vertices. It will be seen that the entries of Φ are closely related to the expansion coefficients of the sets Z_i.

3 Lemmata

The essential idea behind the proof is to compare the eigenvalues of a graph against the eigenvalues of some simple auxiliary graphs from which we can determine easily determine expansion coefficients. We shall complete the proof in several steps by establishing some major lemmata from which the result follows quickly.

The first lemma shows that the eigenvalues of the linear operator $\Phi_G(\mathscr{Z})$ are always smaller than the eigenvalues of the graph G, so long as the sets comprising $\mathscr{Z}$ are disjoint. This is in consequence of the fact that we are restricting the dimension of the problem from $|V|$ to $|\mathscr{Z}|$.

Lemma 3.1 *Let $G = (V, E)$ be an undirected graph. Let $\mathscr{Z} = \{Z_0, \ldots, Z_t\}$ be a sequence of nonempty, disjoint subsets of V. Then the eigenvalues of the linear operator $\Phi = \Phi_G(\mathscr{Z})$ satisfy the relation $\lambda_i(\Phi) \leq \lambda_i(G)$ for each $i = 0, 1, \ldots, t$.*

Proof. Let us define a weighted inner product $\langle \cdot, \cdot \rangle$ on the space $\mathbb{R}^{\mathscr{Z}}$ by the sum

$$\langle r, s \rangle = \sum_{\alpha=0}^{t} |Z_\alpha| \, r(Z_\alpha) \, s(Z_\alpha).$$

And define a linear function $\varphi : \mathbb{R}^{\mathscr{Z}} \to \ell^2(V)$ by the map $\varphi : \chi_{\{Z_i\}} \mapsto \chi_{Z_i}$. It is easy to check that the rank of φ equals t by observing that the vectors $\{\chi_{Z_i}\} \subseteq \ell^2(V)$ are independent.

Now, we prove that φ preserves the inner product, viz.

$$\varphi(r) \cdot \varphi(s) = \langle r, s \rangle$$

for every $r, s \in \mathbb{R}^{\mathscr{Z}}$. Since both sides of the equation are bilinear, we need only check that the formula holds on a basis. So we calculate

$$\varphi(\chi_{\{Z_i\}}) \cdot \varphi(\chi_{\{Z_j\}}) = \chi_{Z_i} \cdot \chi_{Z_j} = \delta_{ij} \, |Z_i|$$

using the fact that the sets $\{Z_i\}$ are disjoint. We may reinterpret this last term as $\langle \chi_{\{Z_i\}}, \chi_{\{Z_j\}} \rangle$, so we have proven the claim.

Next, we establish that for each $r, s \in \mathbb{R}^{\mathscr{Z}}$ we have

$$\varphi(r) \cdot A\varphi(s) = \langle r, \Phi s \rangle.$$

As before, we need only check the basis. We compute

$$\begin{aligned}
\varphi(\chi_{\{Z_i\}}) \cdot A\varphi(\chi_{\{Z_j\}}) &= \chi_{Z_i} \cdot A\chi_{Z_j} \\
&= \sum_{(v,w)\in E} \chi_{Z_i}(v)\,\chi_{Z_j}(w) \\
&= e(Z_i, Z_j) \\
&= \sum_{\alpha=0}^{t} |Z_\alpha|\,\chi_{\{Z_i\}}(Z_\alpha)\,\frac{e(Z_\alpha, Z_j)}{|Z_\alpha|} \\
&= \sum_{\alpha=0}^{t} |Z_\alpha|\,\chi_{\{Z_i\}}(Z_\alpha)\,(\Phi\chi_{\{Z_j\}})(Z_\alpha) \\
&= \langle \chi_{\{Z_i\}}, \chi_{\{Z_j\}} \rangle.
\end{aligned}$$

In particular, we see that

$$\langle r, \Phi s\rangle = \varphi(r) \cdot A\varphi(s) = A\varphi(r) \cdot \varphi(s) = \langle \Phi r, s\rangle.$$

Therefore, Φ is self-adjoint with respect the inner product $\langle \cdot, \cdot \rangle$. It follows that Φ has real eigenvalues.

We may express the eigenvalues of Φ using the Rayleigh quotient:

$$\lambda_i(\Phi) = \max_{R\subseteq \mathbb{R}^{\mathscr{Z}}} \min_{r\in R\setminus\{0\}} \frac{\langle r, \Phi r\rangle}{\langle r, r\rangle}$$

where R ranges over all $(i+1)$-dimensional subspaces of $\mathbb{R}^{\mathscr{Z}}$. Next, we rewrite the $\mathbb{R}^{\mathscr{Z}}$ inner product in terms of the $\ell^2(V)$ inner product and apply the fact that φ has full column rank.

$$\begin{aligned}
\lambda_i(\Phi) &= \max_{R\subseteq \mathbb{R}^{\mathscr{Z}}} \min_{f\in\varphi(R)\setminus\{0\}} \frac{f \cdot Af}{f \cdot f} \\
&\leq \max_{L\subseteq \ell^2(V)} \min_{f\in L\setminus\{0\}} \frac{f \cdot Af}{f \cdot f} \\
&= \lambda_i(A(G))
\end{aligned}$$

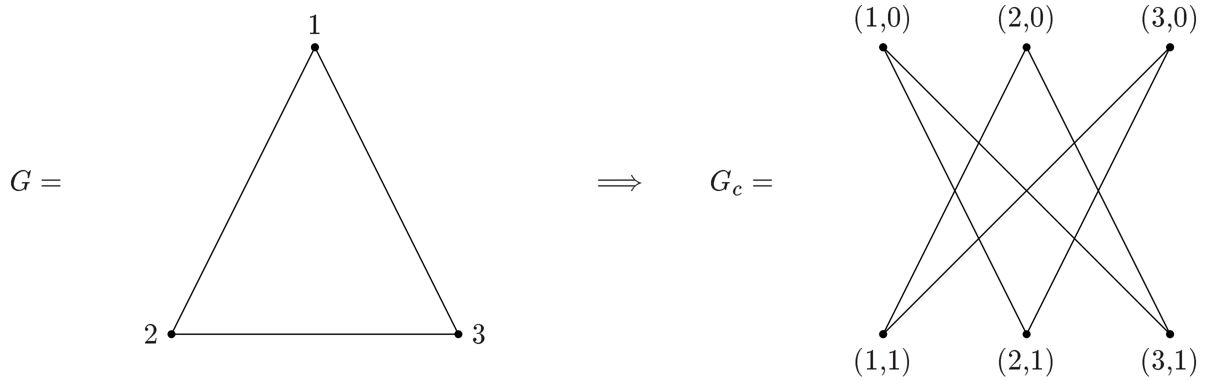
where L ranges over all $(i+1)$-dimensional subspaces of $\ell^2(V)$. The inequality follows because $\dim \mathbb{R}^{\mathscr{Z}} \leq \dim \ell^2(V)$, which is due to the sets in $\mathscr{Z}$ being disjoint. □

As we have mentioned, the expansion coefficient of a set Z is the ratio $|\partial Z|/|Z|$. If a graph is k-regular, then it it easy to see that $e(Z, \partial Z) = k|Z|$. Therefore, we have the relation

$$\frac{|\partial Z|}{|Z|} = k\left(\frac{e(Z, \partial Z)}{|\partial Z|}\right)^{-1}.$$

If we put $\mathscr{Z} = \{Z, \partial Z\}$, then we discover that $\Phi_G(\mathscr{Z})$ contains an entry which is inversely proportional to the expansion coefficient of Z. But we cannot necessarily apply the first lemma in this situation, since Z and ∂Z may not be disjoint. As a result, we shall consider an auxiliary graph where we can ensure that a set and its neighbors do not intersect.

For any graph G, we introduce the *cover graph* G_c which is defined as the tensor product of G and the 2-cycle K_2. The cover graph is a bipartite graph which contains two copies of the vertex set of G. Each vertex in G_c is connected to the same vertices as in G but on the opposite copy of G. Formally, $G_c = (V_c, E_c)$ where $V_c = V \times \{0, 1\}$ and $((u, i), (v, j)) \in E_c$ if and only if $(u, v) \in E$ and $i \neq j$. As an example, we display the cover graph of the 3-cycle:



As a result of this construction, if Z is a subset of $V \times \{0\}$ then ∂Z is a subset of $V \times \{1\}$ and vice-versa. Thus $Z \cap \partial Z = \emptyset$ whenever Z is contained on one side of the cover graph. Another advantage of using the cover graph is that the eigenvalues of G_c are $\pm\lambda_i(G)$. Therefore, $\lambda_1(G_c)$ is always the second largest eigenvalue of G in absolute value, even if this quantity is negative.

In the second lemma, we shall construct a sequence of disjoint sets $Z_i \subseteq G_c$ for which the corresponding linear operator $\Phi_{G_c}(\mathscr{Z})$ has a particularly simple form. Moreover, the nonzero elements of Φ will be closely related to the expansion of the sets in the sequence. Later, we shall estimate these elements in terms of the eigenvalues of Φ, which we shall compute explicitly.

Lemma 3.2 *Let $G = (V, E)$ be an undirected graph. Let $\{X_0, \ldots, X_t\}$ be a sequence of nonempty subsets of V which satisfy the following three conditions.*

1. *Every vertex in each set must have degree equal to k;*
2. *the neighbors of each set must be contained in its successor: $\partial X_i \subseteq X_{i+1}$; and*
3. *the size of the sets are strictly increasing: $|X_i| < |X_{i+1}|$.*

Define X_{-1} to be the empty set.

Now, let $\delta_0, \ldots, \delta_t$ be the $t+1$ largest eigenvalues of G in absolute value, numbered so that $|\delta_0| \geq |\delta_1| \geq \cdots \geq |\delta_t|$. Then $\lambda_i(M_{t+1}) \leq |\delta_i|$ for each $i = 0, 1, \ldots, t$, where

$$M_{t+1}(k; \rho_0, \ldots, \rho_t) = \begin{pmatrix} 0 & k & 0 & 0 & \ldots & 0 \\ \rho_0 & 0 & k-\rho_0 & 0 & \ldots & 0 \\ 0 & \rho_1 & 0 & k-\rho_1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \rho_{t-2} & 0 & k-\rho_{t-2} \\ 0 & \ldots & 0 & 0 & \rho_{t-1} & 0 \end{pmatrix}.$$

The subscript $t+1$ indicates the size of the matrix. Here and throughout, we use the shorthand

$$\rho_i = k\frac{|X_i| - |X_{i-1}|}{|X_{i+1}| - |X_{i-1}|}.$$

Observe that $\rho_0 = k|X_0|/|X_1|$. If $X_1 = \partial X_0$, we find that ρ_0 is inversely proportional to the expansion coefficient of X_0. Indeed, $\partial X_0 \subseteq X_1$ implies that

$$\rho_0 \geq k(|\partial X_0|/|X_0|)^{-1}.$$

This fact suggests what role M_{t+1} will play in the proof of the main theorem. At this point, it is also worth noting that each $\rho_i < k$ since the sets X_i are increasing in size. As a result, all the entries of M_{t+1} are positive, and the ℓ^1-norm of each row is exactly k.

Proof. Let G_c be the cover graph of G. Since $G_c = G \otimes K_2$, the adjacency matrix $A(G_c) = A(G) \otimes A(K_2)$. Therefore, the eigenvalues of G_c satisfy $\lambda_i(G_c) = |\delta_i|$ for $i = 0, 1, \dots, t$.

For each $i = 0, 1, \dots, t$, define a subset of the cover graph $Y_i \subseteq V_c$ by $Y_i = X_i \times \{i \bmod 2\}$. So these sets Y_i are equal to the sets X_i, but they alternate between the two parts of the cover graph. In consequence, the size of Y_i equals the size of the corresponding X_i; every vertex in Y_i has degree k; the neighbors of Y_i are contained in Y_{i+1}; no neighbor of Y_i is an element of Y_i; and each set satisfies $Y_i \subsetneqq Y_{i+2}$. This last proposition follows from the chain of inclusions $X_i \subseteq \partial(\partial X_i) \subseteq \partial(X_{i+1}) \subseteq X_{i+2}$, which is due to the fact that G is undirected and the inequalities $|X_i| < |X_{i+1}|$.

Next, put $Z_i = Y_i \setminus Y_{i-2}$, where we have the convention that $Y_{-1} = Y_{-2} = Y_{-3} = \emptyset$. These sets Z_i are nonempty and disjoint because $\{Y_{2i}\}$ and $\{Y_{2i+1}\}$ both form strictly increasing sequences. For the same reason, $|Z_i| = |Y_i| - |Y_{i-2}| = |X_i| - |X_{i-2}|$. Essentially, these Z_i contain the additional elements in X_i which were not included in X_{i-2}. Since each X_i contains all the neighbors of the previous set X_{i-1}, the Z_i are closely related to the marginal expansion of the X_i.

Now, we set $\mathscr{Z} = \{Z_0, \dots, Z_t\}$ and apply the first lemma to obtain the inequalities $\lambda_i(\Phi_{G_c}(\mathscr{Z})) \le \lambda_i(G_c) = |\delta_i|$ for $i = 0, 1, \dots, t$.

It remains to prove that $\Phi = \Phi_{G_c}(\mathscr{Z}) = M_{t+1}$. First, we shall demonstrate that most of the entries of Φ are zero. For each i, we have the chain of inclusions

$$\partial Z_i = \partial(Y_i \setminus Y_{i-2}) \subseteq \partial Y_i \subseteq Y_{i+1} = (Y_{i+1} \setminus Y_{i-1}) \cup Y_{i-1} = Z_{i+1} \cup Y_{i-1}.$$

Now, $\partial Y_{i-3} \subseteq Y_{i-2}$, so we discover that $e(Z_i, Y_{i-3}) = e(Y_i \setminus Y_{i-2}, Y_{i-3}) = 0$. Since no edges lead from Z_i into Y_{i-3}, the set Z_i has no neighbors in Y_{i-3}. So we may subtract the elements of Y_{i-3} to obtain the inclusion $\partial Z_i \subseteq Z_{i+1} \cup Z_{i-1}$. The sets Z_i are disjoint, so that $e(Z_i, Z_j) = 0$ unless $j = i \pm 1$. In consequence, $\Phi_{ij} = 0$ unless $j = i \pm 1$.

It follows that we can compute the entire matrix Φ by determining the quantities $e(Z_i, Z_{i+1})$. The calculation is fairly straightforward.

$$\begin{aligned} e(Z_i, Z_{i+1}) &= e(Y_i \setminus Y_{i-2}, Y_{i+1} \setminus Y_{i-1}) \\ &= e(Y_i, Y_{i+1} \setminus Y_{i-1}) - e(Y_{i-2}, Y_{i+1} \setminus Y_{i-1}) \\ &= e(Y_i, Y_{i+1} \setminus Y_{i-1}) \end{aligned}$$

because $\partial Y_{i-2} \subseteq Y_{i-1}$. Continuing the equation,

$$\begin{aligned} e(Z_i, Z_{i+1}) &= e(Y_i, Y_{i+1}) - e(Y_i, Y_{i-1}) \\ &= k|Y_i| - k|Y_{i-1}| \\ &= k(|X_i| - |X_{i-1}|) \end{aligned}$$

since the vertices in each Y_i have degree k and since every edge extending from Y_i terminates in Y_{i+1}.

Finally, the entries on the subdiagonal of Φ are given by

$$\frac{e(Z_i, Z_{i-1})}{|Z_i|} = \frac{k(|X_{i-1}| - |X_{i-2}|)}{|X_i| - |X_{i-2}|} = \rho_{i-1}.$$

The entries on the superdiagonal of Φ are

$$\begin{aligned}\frac{e(Z_i, Z_{i+1})}{|Z_i|} &= \frac{k(|X_i| - |X_{i-1}|)}{|X_i| - |X_{i-2}|} \\ &= k\frac{(|X_i| - |X_{i-2}|) + (|X_{i-2}| - |X_{i-1}|)}{|X_i| - |X_{i-2}|} \\ &= k - \rho_{i-2}.\end{aligned}$$

After noting that $\rho_{-1} = 0$, we conclude that $\Phi = M_{t+1}$. □

As we have mentioned, ρ_0 is inversely proportional to the expansion coefficient of X_0. Therefore, an upper bound on ρ_0 will allow us to determine a lower bound on the expansion. The next lemma shows how to estimate ρ_0 in terms of the top eigenvalue of the matrix M in a particularly simple case. The proof involves some detailed analysis, so we shall omit it.

Lemma 3.3 *Let ℓ be a positive integer. Given the inequality*

$$\lambda_0(M_{\ell+1}(k; \rho_0, \underbrace{1, \ldots, 1}_{\ell-1 \text{ times}})) \leq 2\sqrt{k-1}\cosh\theta$$

for $\theta \geq 0$, we can bound the coefficient ρ_0 in terms of θ:

$$\rho_0 \leq (1 + e^{2\theta})(1 + O(\ell^{-1})).$$

Note that this bound depends inversely on ℓ. Therefore, we must be able to make ℓ small to reach a good estimate on ρ_0. Unfortunately, the top eigenvalue of a matrix often grows rapidly as the matrix grows larger. We expect the righthand side of the inequality to grow roughly as fast as the top eigenvalue of $M_{\ell+1}$, so the result is useless unless we can obtain a good bound on the top eigenvalue of $M_{\ell+1}$. In the next two lemmata, we shall develop such a bound.

The next result, which comes from another paper of Kahale, provides an upper bound on the top eigenvalue of an induced subgraph of a k-regular graph in terms of the second largest eigenvalue of the supergraph. We shall not prove this fact.

Lemma 3.4 *Let W be a subset of a k-regular graph $G = (V, E)$ and let H be the subgraph of G induced by W. Then we have the inequality*

$$\lambda_0(H) \leq \lambda_1(G) + (k - \lambda_1(G))\frac{|W|}{|V|}.$$

We are finally ready to make the last estimate we need.

Lemma 3.5 *Let $G = (V, E)$ be a k-regular, undirected, connected graph. Let X_0 and X_1 be nonempty subsets of V which satisfy the condition $\partial X_0 \subseteq X_1$. Then for every $\ell \geq 1$, we have the inequality*

$$\lambda_0(M_{\ell+1}(k; \rho_0, \underbrace{1, \ldots, 1}_{\ell-1 \text{ times}})) \leq \lambda_1(G) + 4k^\ell\frac{|X_1|}{|V|}.$$

This bound may look troublesome because of the exponential dependence on ℓ. But we are interested in the case where $|X_1|/|V|$ is extremely small. As a result the second term on the righthand side becomes almost negligible. Comparing this estimate with the estimate for ρ_0, notice that we must balance two opposing concerns. Here, we need to ensure that ℓ is relatively small, whereas our estimate of ρ_0 improves as ℓ increases.

Proof. If $(\lambda, \mathbf{x})$ is an eigenpair of $A(M_{\ell+1})$, we use the Hölder inequality to see that

$$|\lambda x_i| = |(A\mathbf{x})_i| = |\sum_j A_{ij} x_j| \leq ||A_i||_{\ell^1} \, ||\mathbf{x}||_{\ell^\infty}$$

for each i. Since each row of A has ℓ^1-norm equal to k, we see that

$$|\lambda| \leq k \, \frac{||\mathbf{x}||_{\ell^\infty}}{|x_i|} \leq k.$$

Therefore, the eigenvalues of A are at most k. In consequence, we may assume without loss of generality that $k^{\ell-1}|X_1| \leq |V|$. If not,

$$k^\ell \, \frac{|X_1|}{|V|} > k \geq \lambda_0(M_{\ell+1}),$$

from which the result follows immediately.

Now, we must construct a sequence of sets $\{X_i\}$ with $\rho_{i-1} = 1$ for $i = 2, 3, \ldots, \ell$ which additionally satisfy the hypotheses of the second lemma. We use induction.

Assume that we have constructed sets X_j where $j = 2, 3, \ldots, i$ for which $\partial X_{j-1} \subseteq X_j$ and $|X_{j-1}| < |X_j|$. Note that the other hypothesis on vertex degree is trivially satisfied since the graph is k-regular. We must demonstrate the existence of the set X_{i+1}.

First, we must check that there are enough vertices remaining so that $\rho_i = 1$ is possible. Since $\rho_{j-1} = 1$ for each j, we have $|X_{j+1}| = k|X_j| - (k-1)|X_{j-1}| \leq k|X_j|$ for each j. Combining these inequalities, we see that $|X_i| \leq k^{i-1}|X_1|$. Then, by assumption, $|X_i| \leq |V|/k$. Using this fact,

$$|V \setminus X_{i-1}| = |V| - |X_{i-1}| \geq k|X_i| - |X_{i-1}| \geq k(|X_i| - X_{i-1}|).$$

On the other hand, $e(X_i, V \setminus X_{i-1}) = e(X_i, V) - e(X_i, X_{i-1}) = k(|X_i| - |X_{i-1}|)$. So

$$|\partial X_i \setminus X_{i-1}| \leq e(X_i, V \setminus X_{i-1}) = k(|X_i| - |X_{i-1}|).$$

We conclude that there exists a set X_{i+1} which contains ∂X_i and which satisfies the equation

$$|X_{i+1} \setminus X_{i-1}| = k(|X_i| - |X_{i-1}|).$$

It follows that $\rho_i = 1$, so long as $|X_i| > |X_{i-1}|$. In the theory of k-regular, connected graphs, it is proven that $|\partial X| > |X|$ when $|X| < |V|/2$. Therefore, $|X_i| > |X_{i-1}|$. (By the same virtue, $|X_1| > |X_0|$.

Let H be the subgraph of G induced by $W = X_{\ell-1} \cup X_\ell$. We apply the second lemma to H and the sets $X_0, \ldots, X_\ell \subset H$. Since $A(H)$ has no negative entries, classical linear algebra shows that $\lambda_0(H)$ is the largest eigenvalue of H in absolute value. Therefore, $\lambda_0(M_{\ell+1}(k; \rho_0, 1, \ldots, 1)) \leq \lambda_0(H)$. On the other hand, the fourth lemma provides an upper bound on $\lambda_0(H)$:

$$\lambda_0(H) \leq \lambda_1(G) + (k - \lambda_1(G)) \frac{|W|}{|V|}.$$

The second eigenvalue of G satisfies $\lambda_1(G) < k$, and the foregoing arguments show that $|W| \leq |X_{\ell-1}| + |X_\ell| \leq 2k^{\ell-1}|X_1|$. So we may conclude algebraically that

$$\lambda_0(M_{\ell+1}(k;\rho_1,1,\ldots,1)) \leq \lambda_1(G) + 4k^\ell \frac{|X_1|}{|V|}.$$

□

4 The Main Theorem

Finally, we combine these results to obtain the desired lower bound. The proof is a fairly straightforward application of the lemmata.

Theorem 4.1 *Let $G = (V, E)$ be a k-regular, undirected graph. Put $\lambda = \max\{\lambda_1(G), 2\sqrt{k-1}\}$. Then, for any nonempty subset X of G which satisfies $|X| \leq k^{-1/\varepsilon}|V|$, we may bound the expansion coefficient of X below using*

$$\frac{\partial X}{X} \geq \frac{k}{2}\left(1 - \sqrt{1 - \frac{4(k-1)}{\lambda^2}}\right)(1 + O(\varepsilon)).$$

Proof. We have chosen λ to be large enough that we can write it in the form $\lambda = 2\sqrt{k-1}\cosh\theta$ for some $\theta \geq 0$. We choose $X_0 = X$ and $X_1 = \partial X$. Apply the final lemma to these subsets using $\ell = \lfloor (2\varepsilon)^{-1} \rfloor$. We obtain the inequality

$$\lambda_0(M_{\ell+1}(k;\rho_0,1,\ldots,1)) \leq \lambda_1(G) + 4k^\ell \frac{|\partial X|}{|X|} \equiv \tilde{\lambda},$$

where we take the righthand side of the inequality as the definition of $\tilde{\lambda}$.

Let us estimate the difference between $\tilde{\theta}$ and θ. It is clear that $\tilde{\lambda}$ exceeds λ, so we may express $\tilde{\lambda} = 2\sqrt{k-1}\cosh\tilde{\theta}$ for some $\tilde{\theta} \geq \theta$. Since $|\partial X| \leq k|X| \leq k^{1-1/\varepsilon}|V|$, we discover that $k^\ell|\partial X|/|V| \leq k^{1-1/(2\varepsilon)}$. Therefore,

$$\tilde{\lambda} - \lambda \leq 4k \cdot k^{-1/(2\varepsilon)} = O(\varepsilon^2),$$

where the asymptotic estimate holds because the decay is exponential in ε. Thus, $\tilde{\lambda} = \lambda(1 + O(\varepsilon))$ and $\cosh\tilde{\theta} - \cosh\theta = O(\varepsilon^2)$. Applying the inequality $(x-y)^2 \leq 2|\cosh x - \cosh y|$, we obtain $\tilde{\theta} - \theta = O(\varepsilon)$.

Now, since $\tilde{\lambda}$ is an upper bound on the top eigenvalue of $M_{\ell+1}$, we may use the third lemma to estimate ρ_0 in terms of $\tilde{\theta}$. Then, we apply our asymptotics to express the result in terms of θ.

$$\rho_0 \leq (1 + e^{2\tilde{\theta}})(1 + O(\ell^{-1})) = (1 + e^{2\tilde{\theta}})(1 + O(\varepsilon)) = (1 + e^{2\theta})(1 + O(\varepsilon))$$

since a first order change in x near zero yields a first-order change in e^x: $e^{O(\varepsilon)} = 1 + O(\varepsilon)$.

As we have discussed, $\rho_0 = k|X|/|\partial X|$. Therefore,

$$\frac{|\partial X|}{|X|} = \frac{k}{\rho_0} \geq \frac{k}{2e^\theta\cosh\theta}(1 + O(\varepsilon))$$

using the definition of the hyperbolic cosine. Applying standard identities, $e^{-\theta} = \cosh\theta - \sinh\theta = \cosh\theta - \sqrt{\cosh^2\theta - 1}$. In consequence, the definition of θ yields

$$\frac{1}{e^{\theta}\cosh\theta} = 1 - \sqrt{1 - \frac{1}{\cosh^2\theta}} = 1 - \sqrt{1 - \frac{4(k-1)}{\lambda^2}}.$$

We conclude that

$$\frac{|\partial X|}{|X|} \geq \frac{k}{2}\left(1 - \sqrt{1 - \frac{4(k-1)}{\lambda^2}}\right)(1 + O(\varepsilon)).$$

$\square$