CS395T Pseudorandomness and Combinatorial Constructions
Project Presentation, May 1, 2001
Ramsey Graphs
presented by: K. J. Balakrishnan

1 Introduction

Ramsey Graph Theory deals with the regularity and coloring of graphs. A graph is called p-ramsey if it contains a complete monochromatic sub-graph on p vertices [5]. Any graph on n vertices can be seen as a two-coloring of the complete graph on n vertices - one color if there is an edge between two vertices and another if there is no edge between those two vertices in the given graph. In this presentation, I first show the construction of a graph on n vertices with no clique or independent set of size t such that

$$n = t^{\frac{\sqrt{\log\log t}}{\log\log\log t}}$$

In the second part, I talk about a theorem by *Zs. Nagy* which deals with a limit on the number of vertices of a graph that does not contain a monochromatic subgraph on (t+1) vertices.

2 Construction of Ramsey Graphs

Ramsey showed that every graph on N nodes must contain a clique or independent set of size $\frac{1}{2}\log N$. Later Erdos showed using probabilistic methods that there are graphs on N nodes that don't contain a clique or independent set on $2\log N$ vertices [4]. But no explicit construction was given and it is still an open problem to explicitly construct graphs with that limit. In fact, the largest graph that can be explicitly constructed with no clique or independent set of size t was given by Frankl [6] to be of size $t^{\frac{\log t}{\log\log t}}$. In this section, an elementary construction of such graphs is shown with size superpolynomial in t [2].

Definition 1 *Superpolynomial Function: A function that grows faster than any polynomial of the variable. i.e.* $f(t) = t^{g(t)}$ *where* $\lim_{t\to\infty} g(t) = \infty$.

The main idea behing this construction comes from a similar result in error correcting codes in which good error correcting codes are obtained by concatenating a good code over a large alphabet (which can be explicitly constructed) with good codes that map the large alphabet to a small one. Hence we combine many small graphs, most of which are known to be good for our purpose to get our graph. For this we need to define the product of graphs.

Definition 2 *Product of Graphs: Let graphs* G_1 *and* G_2 *be graphs on* n_1 *and* n_2 *nodes respectively and do not contain cliques or independent sets of size* k_1 *and* k_2 *respectively. Then, their product is a graph on* $n_1 \cdot n_2$ *nodes that doesn't contain any clique or independent set of size* $k_1 \cdot k_2$ *defined as:* (u_1, u_2) *is connected to* (v_1, v_2) *iff*

- $u_1 \neq v_1$ *and they are connected in* G_1.
- $u_1 = v_1$ *and* u_2 *is connected to* v_2 *in* G_2.

Corollary 1 *If we have a collection of graphs $G_1, G_2, \ldots G_m$ of size $n_1, n_2 \ldots n_m$ and numbers $k_1, k_2 \ldots k_m$ such that G_i does not contain a clique or independent set of size k_i, then H = product of $G_1, G_2, \ldots G_m$ is of size $\Pi_{i=1}^m n_i$ and does not contain a clique or independent set of size $\Pi_{i=1}^m k_i$.*

Let D be a collection of m graphs on n nodes such that

$$\Pr[\text{any graph } G \text{ in } D \text{ has a clique or independent set of size } k] \leq \alpha.$$

Note that for αm graphs in D which have cliques or independent sets of size $\geq k$, we don't assume anything, i.e. they may have cliques or independent sets of size n.

Lemma 1 *Consider H = product of all graphs in D. The number of nodes $N = n^m$ and there are no cliques or independent sets of size $t = k^{(1-\alpha)m} n^{\alpha m}$. If $\alpha < \frac{1}{\log n}$, then $t < (2k)^m$.*

To apply the lemma, we need to come up with k and n with $k = O(\log n)$ and as small a collection D as possible with $\alpha < \frac{1}{\log n}$.
Example: Let D be the set of all graphs on n nodes and $k = 2\log n$. For this collection $\alpha < \frac{1}{\log n}$. Here H is on $n^{2^{\binom{n}{2}}}$ nodes and does not have a clique or independent set of size $t = (2\log n).^{2^{\binom{n}{2}}}$ Therefore,

$$N \geq t^{\frac{\log\log\log t}{\log\log\log\log t}}$$

We can do better by taking small probability spaces.

2.1 A construction based on small biased probability spaces

Recall the definition of k-wise ϵ-biased random variables.

Definition 3 *Random Variables $X_1, X_2 \ldots X_n \in \{0,1\}$ are k-wise ϵ-biased if*

$$|\Pr[X_{i_1} \oplus \ldots \oplus X_{i_k} = 0] - \Pr[X_{i_1} \oplus \ldots \oplus X_{i_k} = 1]| \leq \epsilon$$

$$i_1, i_2 \ldots i_k \in [n].$$

Since the variation distance of this distribution from the uniform distribution is less than $2^{k/2}\epsilon$, we have

$$\Pr[X_{i_1} X_{i_2} \ldots X_{i_k} = 0, 1] \leq 2^{-k+1} + 2^{k/2}\epsilon.$$

Construction: If we choose edges of our graph on n vertices from a $\binom{k}{2}$-wise ϵ-biased probability space on $\binom{n}{2}$ random variables, then the probability that there exists a clique or independent set of size k is at most

$$\alpha = \binom{n}{k}(2^{-\binom{k}{2}+1} + 2^{\binom{k}{2}/2}\epsilon)$$

Taking $k = 2\log n$ and $\epsilon = 2^{-k^2}$, we can get $\alpha \leq \log n$. Consider now a D defined by this probability space, i.e. every graph corresponds to a point in the probability space. The constructions by Alon [3] are of size $(n/\epsilon)^c$ for some fixed c. Therefore, $m = 2^{c'k^2}$ for some fixed c'. The number of vertices in the product graph, H, will then be $N = 2^{k 2^{c'k^2}}$ and it will not have a clique or independent set of size $t = (2k).^{2^{c'k^2}}$ Or,

$$N \geq t^{\frac{c'\sqrt{\log\log t}}{\log\log\log t}}$$

3 Limits on Ramsey Numbers

In this section, we prove a theorem which the limits the size of the Ramsey graph with no clique or independent set of size greater than t [1]. In order to prove the theorem, we require the following lemma which we first prove.

Lemma 2 *Let $A_i \subset [n]$, $i = 1, 2 \dots m$ such that*

1. $|A_i| = 3, \forall\, i$
2. $|A_i \cap A_j| = 1, \forall\, i \neq j$.

Then $m \leq n$.
Proof: Let v_i be the incidence vector of A_i. Therefore $v_i \in \mathbf{R}^n$. The conditions can now be written in terms of the standard dot product of the incidence vectors as:

1. $v_i \cdot v_i = 3$
2. $v_i \cdot v_j = 1, \forall\, i \neq j$.

We now show that all such v_i's are linearly independent. Let $X = \sum_m \lambda_i v_i = 0$ for some $\lambda_1, \dots \lambda_m$. Let $Y = [\lambda_1 \lambda_2 \dots \lambda_m]^T$. By taking dot product of X with v_i for all $i \in [m]$, we get m equations in $\lambda_1, \lambda_2 \dots \lambda_m$. Write them as $AY = 0$, where

$$A = \begin{bmatrix} 3 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{bmatrix} = J - (-2)I,\ J = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

Now det(A) $= 0$ iff eigenvalue of J is -2. But the eigenvalues of J are m and 0 (of multiplicity $m-1$). Hence the only solution to the equations are $Y = 0$ which means all the v_i's are linearly independent. Since there can be atmost n linearly independent vectors in $\mathbb{R}^n$ space, $m \leq n$. $\square$

Theorem 1 (Zs. Nagy, 1972) *For $v = \binom{t}{3}$, there exists a constructive two coloring of the complete graph on v vertices without a K_{t+1} (a complete graph on $(t+1)$ vertices) in either color.*

Proof: Identify the set of vertices of K_v with the set of triplets from $X = \{1, 2, \dots t\}$.
Construction: Join two such triplets by a red edge if they have precisely one vertex in common and by blue edge otherwise.
Suppose m triplets are pairwise joined by the same color. We have to prove $m \leq t$. If the color is red, $m \leq t$ by Lemma 2. If the color is blue, all pairwise intersections will have size 0 or 2. So $m \leq t$ from the Oddtown Theorem proved below. $\square$

Theorem 2 (Oddtown Theorem) *In a town of n citizens, no more than n clubs can be formed under the folowing rules:*

- Each club has odd number of members.
- Each pair of clubs share an even number of members.

Proof: Suppose we have m clubs, $C_1, C_2 \ldots C_m$. Let v_i be the incidence vector of the C_i with respect to the citizens. $v_i \in \mathbb{R}^n$, the j^{th} entry of v_i is 1 or 0 according to whether or not citizen $\#i$ belongs to the club C_i. Now, $v_i \cdot v_j = |C_i \cap C_j|$. The rules can now be written as

$$v_i \cdot v_j = \text{odd},\ i = j$$
$$v_i \cdot v_j = \text{even},\ i \neq j$$

If we simplify further by working only over $\mathbb{F}_2$, the rules reduce to

$$v_i \cdot v_j = 1,\ i = j$$
$$v_i \cdot v_j = 0,\ i \neq j$$

Under these circumstances, the v_i's are linearly independent over $\mathbb{F}_2$. Let $\lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_m v_m = 0$, $(\lambda_i \in \mathbb{F}_2)$. Taking dot product w.r.t v_1 gives $\lambda_1 = 0$. Similarly taking dot products with each v_i, we get that all λ_i are 0. Hence v_i are linearly independent, but since there can only be n linearly independent vectors in $\mathbb{R}^n$, we have $m \leq n$. $\square$

References

[1] L. Babai and P. Frankl, Linear Algebra Methods in Combinatorics, preliminary version manuscript, 1992.

[2] M. Naor, Constructing Ramsey Graphs from small probability spaces, *IBM Research Report on Computer Science*, RJ 8810 (70940), June 2, 1992.

[3] N. Alon, O. Goldreich, J. Hastad and R. Peralta, Simple Constructions for almost k-wise independent random variables. In *Proc. 31st IEEE Symp. on Foundations of Computer Science*, 1990, pp. 544-553.

[4] P. Erdos, Some remarks on the theory of graphs, *Bulletin of American Mathematics Society*, 53, 292-294, 1947.

[5] R. L. Graham, B. L. Rothschild and J. H. Spencer, Ramsey Theory, John Wiley and Sons Inc., New York, 1980.

[6] P. Frankl, A constructive lower bound for Ramsey numbers, *Ars Combinatorica 3*, 297-302, 1977.