On the Second Eigenvalue and Linear Expansion of Regular Graphs

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Abstract

We investigate the relation between the second eigenvalue and the linear expansion of regular graphs. The spectral method is the best currently known technique to prove lower bounds on the expansion. We improve this technique by showing that the expansion coefficient of linear-sized subsets of a k-regular graph G is at least $\frac{k}{2}\left(1-\sqrt{\max(0,1-\frac{4k-4}{\lambda_1(G)^2})}\right)^-$, where $\lambda_1(G)$ is the second largest eigenvalue of the graph. In particular, the linear expansion of Ramanujan graphs, which have the property that the second largest eigenvalue is at most $2\sqrt{k-1}$, is at least $(k/2)^-$. This improves upon the best previously known lower bound of $3(k-2)/8$. For any integer k such that $k-1$ is prime, we explicitly construct an infinite family of k-regular graphs G_n on n vertices whose linear expansion is $\frac{k}{2}$ and such that $\lambda_1(G_n) \leq 2\sqrt{k-1}+o(1)$. Since the graphs G_n have asymptotically optimal second eigenvalue, this essentially shows that $(k/2)$ is the best bound one can obtain using the second eigenvalue method.

1 Introduction

Given an undirected k-regular graph $G=(V,E)$ and a subset X of V, we define the expansion of X to be the ratio $\frac{|N_G(X)|}{|X|}$, where $N_G(X)=\{w \in V : \exists v \in X,\ (v,w) \in E\}$ is the set of neighbors of X. Graphs whose all subsets of size lying in a given range have large expansion are called expander graphs.

Expander graphs are widely used in Computer Science, in areas ranging from parallel computation [2, 5, 14, 19, 22] to complexity theory and cryptography [1, 6, 10, 23]. The range of the subsets whose expansion is relevant and the magnitude of the expansion needed depends on the nature of the application. For example, in the design of the AKS sorting circuit, we use expanders of constant degree such that subsets of size at most $\epsilon|V|$ have expansion at least $\frac{1-\epsilon}{\epsilon}$, where ϵ is a fixed positive constant. The depth of the resulting network is proportionnal to the degree of the expander. In other applications, like the construction of non-blocking networks in [5], we need a family of fixed degree uneven bipartite expanders where the expansion of linear-sized subsets is greater than $\frac{k}{2}$. Indeed, an expansion greater than $k/2$ guarantees that a constant fraction of any linear-sized subset have *unique neighbors*, which is crucial in the construction in [5].

It is known that random regular graphs are good expanders. For example, for any $\beta < k-1$, there exists a constant α such that, with high probability, all the subsets of a random k-regular graph of size at most αn have expansion at least β. However, the explicit contruction of expander graphs is much more difficult. The first constructions [16, 9, 11] use techniques from group representation theory and harmonic analysis to prove the desired expansion properties.

The best currently known method to calculate lower bounds on the expansion in polynomial time relies on analysing the second eigenvalue of the graph. It is known that all the eigenvalues of the adjacency matrix $A(G)$ of G are real. Let $\lambda_i(G)$ denote the i-th largest eigenvalue of G. We have $\lambda_0(G)=k$ and $\lambda(G)=\max(\lambda_1(G),|\lambda_{n-1}(G)|) \leq k$, with equality iff G is not connected or bipartite. The relation between the expansion and the spectrum of a graph was introduced by Tanner [21] who showed that, for any subset X of a k-regular graph

$$|N_G(X)| \geq \frac{k^2|X|}{\lambda^2+(k^2-\lambda^2)\frac{|X|}{n}} \qquad (1)$$

The smaller λ is, the higher expansion this bound implies. However, since $\liminf \lambda(G_n) \geq 2\sqrt{k-1}$ for any family of k-regular graphs G_n [3], the best asymptotic expansion coefficient one can get by Tanner's result is $\frac{k}{4}+o(k)$. This bound is achieved by Ramanujan graphs. By definition, a Ramanujan graph is a connected k-regular graph whose eigenvalues $\neq \pm k$ are

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at most $2\sqrt{k-1}$ in absolute value. Infinite families of Ramanujan graphs have been explicitly constructed in [15, 17] when $k-1$ is prime. The linear expansion of Ramanujan graphs was recently [12] improved to $3(k-2)/8$.

We define the linear expansion of a family of k-regular graphs G_n on n vertices to be the best lower bound on the expansion of subsets of size up to αn, where α is an arbitrary small positive constant. Our aim is to calculate the best linear expansion one can prove using the second eigenvalue technique. In this paper, we prove that the expansion of linear subsets of a k-regular graph G is at least $\frac{k}{2}\left(1-\sqrt{\max(0, 1-\frac{4k-4}{\lambda_1(G)^2})}\right)^-$. In particular, linear-sized subsets of Ramanujan graphs have expansion at least $\frac{k}{2}^-$. On the other hand, for any integer k such that $k-1$ is a prime congruent to 1 modulo 4, and for any function m of n such that $m = o(n)$, we explicitly construct an infinite family of k-regular graphs G_n on n vertices such that $\lambda(G_n) \le 2\sqrt{k-1}(1+o(1))$ and G_n contains a subset of size $2m$ with expansion $k/2$. Since such a family has asymptotically optimal second eigenvalue, this essentially shows that $k/2$ is the best bound lower bound on the linear expansion one can obtain by the second eigenvalue method. However, it is still an open question whether there exists a family of Ramanujan graphs with linear expansion $k/2$. As a biproduct of our techniques, we obtain known bounds on the number of edges in an induced subgraph of a regular graph, we give a simple proof of Tanner's inequality, and we establish a lower bound of $2\sqrt{k-1}(1+O(\log_k^{-2} n))$ on the second eigenvalue of any k-regular graph. A previous lower bound of $2\sqrt{k-1}(1+O(\log_k^{-1} n))$ was proven in [18] and improved to $2\sqrt{k-1}(1+O(\log_k^{-2} n))$ in [8]. Our results provide an efficient way to test that the expansion of linear sized subsets of random graphs is at least $\frac{k}{2} + O(k^{3/4}\log^{1/2} k)$. As an application of the improved expansion of Ramanujan graphs, we can build explicit selection networks of asymptotic size $(3+\epsilon)n\log_2 n$, for any $\epsilon > 0$, improving on the bound $6n\log_2 n$ that was previously known. As defined in [19], a selection network is a network of comparators that classifies a set of n numbers, where n is even, into two subsets of $n/2$ numbers such that any element in the first set is smaller than any element in the second set.

2 Notation, definitions, and background

Throughout the paper, $G = (V, E)$ will denote an undirected connected graph on a set V of vertices. If G is regular, it is easy to see that $|N_G(X)| \ge |X|$ for any subset X. Let $L^2(V)$ denote the set of real valued functions on V and $L_0^2(V) = \{f \in L^2(V);\ \sum_{v\in V} f(v) = 0\}$. As usual, we define the scalar product of two vectors f and g of $L^2(V)$ by

$$f \cdot g = \sum_{v\in V} f(v)g(v),$$

and the euclidean norm of a vector f by $||f|| = \sqrt{f\cdot f}$. We denote the adjacency matrix of G by $A(G)$, or simply by A if there is no risk of confusion. $A(G)$ is the 0-1 $n\times n$ matrix whose (i,j) entry is equal to 1 iff $(i,j)\in E$. If we consider $f \in L^2(V)$ as a row vector, we have

$$(Af)(v) = \sum_{(v,w)\in E} f(w)$$

A defines a self-adjoint operator since $\forall f, g \in L^2(V)$ we have

$$(Af)\cdot g = f\cdot(Ag) = \sum_{(v,w)\in E} f(v)g(w) \qquad (2)$$

The *girth* of G, denoted by $c(G)$, is the length of the shortest cycle in G. For any subset W of V, we denote by χ_W the characteristic vector of W:

$$\chi_W(v) = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{otherwise} \end{cases}$$

For any matrix M with real eigenvalues, we denote by $\lambda_i(M)$ the i-th largest eigenvalue of M. We also denote $\lambda_i(A(G))$ by $\lambda_i(G)$. If X and Y are two subsets of a graph $G = (V, E)$, then $e(X,Y) = |\{(u,v)\in X\times Y \cap E\}|$. If $Z_0,\ldots,Z_t$ is a sequence of non-empty subsets of V, we denote by $R^{\{Z_0,Z_1,\ldots,Z_t\}}$ the set of real valued functions on $\{Z_0, Z_1, \ldots, Z_t\}$. We define $\Phi(Z_0, Z_1, \ldots, Z_t)$ to be the linear mapping in $R^{\{Z_0,Z_1,\ldots,Z_t\}}$ whose matrix in the canonical basis $(\chi_{\{Z_0\}},\ldots,\chi_{\{Z_t\}})$ is the $(t+1, t+1)$ matrix with entry (i,j) equal to $\frac{e(Z_i,Z_j)}{|Z_i|}$. $\Phi(Z_0, Z_1, \ldots, Z_t)$ can be viewed as the adjacency matrix of the weighted directed graph on $R^{\{Z_0,Z_1,\ldots,Z_t\}}$ where the weight of the edge (Z_i, Z_j) is equal to the average number of neighbors that a node in Z_i has in Z_j.

3 Lower bound on the expansion

Lemma 1 *Let $G = (V, E)$ be a graph and let $Z_0, \ldots, Z_t$ be a sequence of non-empty disjoint subsets of V. For $0 \leq i \leq t$, we have $\lambda_i(G) \geq \lambda_i(\Phi)$, where $\Phi = \Phi(Z_0, Z_1, \ldots, Z_t)$.*

Proof Define the scalar product $<,>$ on $R^{\{Z_0,\ldots,Z_t\}}$ by

$$< r, s > = \sum_{i=0}^{t} |Z_i| r(Z_i) s(Z_i).$$

Let ψ be the linear mapping from $R^{\{Z_0,\ldots,Z_t\}}$ to $L^2(V)$ that mapps $\chi_{\{Z_i\}}$ to χ_{Z_i}.

Claim 1 *For any $r, s \in R^{\{Z_0,\ldots,Z_t\}}$, we have $\psi(r) \cdot \psi(s) = < r, s >$.*

Proof Since both sides of the above equality are bilinear in r and s, it suffices to show that the equality holds when r and s are elements of the canonical basis of $R^{\{Z_0,\ldots,Z_t\}}$. But $\psi(\chi_{\{Z_i\}}) \cdot \psi(\chi_{\{Z_j\}}) = \chi_{Z_i} \cdot \chi_{Z_j} = \delta_{i,j}|Z_i| = < \chi_{\{Z_i\}}, \chi_{\{Z_j\}} >$. ■

Claim 2 *For any $r, s \in R^{\{Z_0,\ldots,Z_t\}}$, we have $\psi(r) \cdot A\psi(s) = < r, \Phi s >$.*

Proof The claim follows from bilinearity and the equalities $\psi(\chi_{\{Z_i\}}) \cdot A\psi(\chi_{\{Z_j\}}) = \chi_{Z_i} \cdot A\chi_{Z_j} = e(Z_i, Z_j) = < \chi_{\{Z_i\}}, \Phi\chi_{\{Z_j\}} >$. ■

This implies in particular that Φ defines a self-adjoint operator with respect to the product $<,>$ and so the eigenvalues of Φ are real. Using the elementary theory of quadratic forms and the injectivity of ψ, it follows that

$$\begin{aligned} \lambda_i(\Phi) &= \max_{L} \min_{r \in L - \{0\}} \frac{< r, \Phi r >}{< r, r >} \\ &= \max_{L} \min_{f \in \psi(L) - \{0\}} \frac{f \cdot Af}{||f||^2} \\ &\leq \max_{L'} \min_{f \in L' - \{0\}} \frac{f \cdot Af}{||f||^2} \\ &= \lambda_i(G), \end{aligned}$$

where L and L' range respectively over the subspaces of $R^{\{Z_0,\ldots,Z_t\}}$ and $L^2(V)$ of dimension $i + 1$. ■

Similarly, one can prove that $\lambda_{n-1-i}(G) \leq \lambda_{t-i}(\Phi)$ for $0 \leq i \leq t$, but we will not need this inequality in the proof of Theorem 1.

Lemma 2 *Let $G = (V, E)$ be a graph and $X_{-1} = \emptyset, X_0, X_1, \ldots, X_t$ be a sequence of subsets of V such that, for $0 \leq i \leq t-1$, the degree of any element of X_i is equal to k, $N_G(X_i) \subseteq X_{i+1}$ and $|X_{i-1}| < |X_i|$. If the eigenvalues of the adjacency matrix of G are $\delta_0, \delta_1, \ldots, \delta_t$, with $|\delta_0| \geq |\delta_1| \geq \cdots \geq |\delta_t|$, then $|\delta_i|$ is greater than or equal to the i-th largest eigenvalue of the matrix $M_{t+1}(k; \rho_0, \rho_1, \ldots, \rho_{t-1})$ equal to*

$$\begin{pmatrix} 0 & k & 0 & 0 & \cdots & 0 \\ \rho_0 & 0 & k-\rho_0 & 0 & \cdots & 0 \\ 0 & \rho_1 & 0 & k-\rho_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \rho_{t-2} & 0 & k-\rho_{t-2} \\ 0 & 0 & \cdots & 0 & \rho_{t-1} & 0 \end{pmatrix},$$

where $\rho_i = k\frac{|X_i| - |X_{i-1}|}{|X_{i+1}| - |X_{i-1}|}$.

Proof Note that $X_i \subseteq N^2(X_i) \subseteq X_{i+2}$. Consider the *cover graph* G_c of G defined on $V_c = V \times \{0, 1\}$ and where $((u, l), (v, m)) \in V_c \times V_c$ is an edge iff $(u, v) \in E$ and $l \neq m$. The adjacency matrix of G_c is the tensor product of $A(G)$ and $A(K_2)$, and so the eigenvalues of G_c are the pairwise products of the eigenvalues of $A(G)$ and $A(K_2)$, namely $\lambda_0, \ldots, \lambda_{n-1}$ and $-\lambda_0, \ldots, -\lambda_{n-1}$, hence $\lambda_i(G_c) = |\delta_i|$ for $0 \leq i \leq n-1$. For $0 \leq i \leq t$, let Y_i be the subset of V_c defined by $Y_i = X_i \times \{(i \bmod 2)\}$. Then, for $0 \leq i \leq t-1$, the degree of any element of Y_i is equal to k, $N_{G_c}(Y_i) \subseteq Y_{i+1}$ and $Y_{i-1} \neq Y_{i+1}$. Now, we apply lemma 1 to G_c and the subsets $Z_i = Y_i - Y_{i-2}$, for $0 \leq i \leq t$. The Z_i's are non-empty and disjoint since Y_i is a strict subset of Y_{i+2} and $Y_i \cap Y_j = \emptyset$ if $i \not\equiv j \pmod 2$. Note that $|Y_i| = |X_i|$ and $|Z_i| = |X_i| - |X_{i-2}|$. For $0 \leq i \leq t-1$, we have $e(Z_i, Z_{i+1}) = e(Y_i - Y_{i-2}, Y_{i+1} - Y_{i-1}) = e(Y_i, Y_{i+1} - Y_{i-1}) - e(Y_{i-2}, Y_{i+1} - Y_{i-1})$. But $e(Y_{i-2}, Y_{i+1} - Y_{i-1}) = 0$ since $N(Y_{i-2}) \subseteq Y_{i-1}$. So $e(Z_i, Z_{i+1}) = e(Y_i, Y_{i+1}) - e(Y_i, Y_{i-1}) = k|Y_i| - k|Y_{i-1}| = k(|X_i| - |X_{i-1}|)$. Finally, $N(Z_i) \subseteq N(Y_i) \subseteq Y_{i+1} = Z_{i+1} \cup Y_{i-1}$. On the other hand, $N(Y_{i-3}) \subseteq Y_{i-2}$ and so there are no edges between Z_i and Y_{i-3} (by convention $Y_{-2} = Y_{-3} = \emptyset$.) Hence $N(Z_i) \subseteq Z_{i+1} \cup Z_{i-1}$ and $e(Z_i, Z_j) = 0$ if $|i - j| \neq 1$. Therefore $\Phi_{G_c}(Z_0, Z_1, \ldots, Z_t) = M_{t+1}(k; \rho_0, \rho_1, \ldots, \rho_{t-1})$. This concludes the proof. ■

Lemma 3 *[12] Let W be a subset of a k-regular graph G and H the subgraph induced on W. Then $\lambda_0(H) \leq \lambda_1(G) + (k - \lambda_1(G))\frac{|W|}{n}$.*

Lemma 4 *Let $G = (V, E)$ be a k-regular connected graph and $X_0, X_1, \ldots, X_t$ a sequence of non-empty* *subsets of V such that, for* $0 \le i \le t-1$, *we have* $N_G(X_i) \subseteq X_{i+1}$. *For* $l \ge 1$, $\lambda_1(G) + 4k^l \frac{|X_t|}{n}$ *is greater than the largest eigenvalue of the matrix* $M_{t+l}(k; \rho_0, \rho_1, \ldots, \rho_{t-1}, \underbrace{1, \ldots, 1}_{l-1 \text{ times}})$.

Proof Since the largest eigenvalue of the matrix M_{t+l} is at most k, we can assume without loss of generality that $|X_t|k^{l-1} \le n$. We recursively construct subsets X_i of G, for $t < i \le t+l-1$, such that $N_G(X_{i-1}) \subseteq X_i$ and $\rho_{i-1} = 1$. Let $i \in \{t, \ldots, t+l-2\}$ and assume that we constructed $X_{t+1}, \ldots, X_i$ such that $\rho_j = 1$ for $t \le j < i$. The condition $\rho_j = 1$ implies that $|X_{j+1}| = k|X_j| - (k-1)|X_{j-1}| \le k|X_j|$, and so $|X_i| \le k^{i-t}|X_t| \le \frac{n}{k}$. On the other hand, $e(X_i, V - X_{i-1}) = e(X_i, V) - e(X_i, X_{i-1}) = k|X_i| - k|X_{i-1}|$ since G is k-regular and $N_G(X_{i-1}) \subseteq X_i$. Therefore, $|N_G(X_i) - X_{i-1}| \le k|X_i| - k|X_{i-1}|$. But $|V - X_{i-1}| = n - |X_{i-1}| \ge k|X_i| - k|X_{i-1}|$. Therefore, there exists a subset X_{i+1} of V containing $N_G(X_i)$ and such that $|X_{i+1} - X_{i-1}| = k|X_i| - k|X_{i-1}|$. Since G is k-regular connected and since $N_G(X_{i-1}) \subseteq X_i$ and $|X_i| < n/2$, we have $|X_{i-1}| < |X_i|$ and so ρ_i is well defined and equal to 1.

Similarly, $|X_{j-1}| < |X_j|$ for $0 \le j < t$ (we define X_{-1} to be the empty set). Now, we apply lemma 2 to the induced graph H on $W = X_{t+l-2} \cup X_{t+l-1}$ and the subsets $X_0, \ldots, X_{t+l-1}$. From the theory of matrices with non-negative entries [20], we know that $\lambda_0(H)$ is the largest eigenvalue of H in absolute value. Hence, $\lambda_0(H) \ge \lambda_0(M_{t+l}(k; \rho_0, \rho_1, \ldots, \rho_{t-1}, 1, \ldots, 1))$. By lemma 3, however, $\lambda_0(H) \le \lambda_1(G) + (k - \lambda_1(G))\frac{|W|}{n} < \lambda_1(G) + 4\frac{|X_t|k^l}{n}$ since $|\lambda_1| < k$ and $|W| \le 2k^{l-1}|X_t|$. ∎

Lemma 5 *The eigenvalues of the matrix* $M_{l+1}(k; \rho_0, 1, \ldots, 1)$ *are* $2\sqrt{k-1}\cosh\theta$, *where* θ *ranges over the solutions (in the complex domain) to the equation*

$$((4k-4)\cosh^2\theta - k\rho_0)s_l(\theta) = 2(k-\rho_0)\cosh(\theta)s_{l-1}(\theta),$$

and $s_i(\theta)$ *is the analytical extension of the function* $\frac{\sinh(i\theta)}{\sinh(\theta)}$ *over the domain of complex numbers. If the largest eigenvalue of* $M_{l+1}(k; \rho_0, 1, \ldots, 1)$ *is at most* $2\sqrt{k-1}\cosh(\theta')$, *with* $\theta' \ge 0$, *then* $\rho_0 \le (1+e^{2\theta'})(1+O(\frac{1}{l}))$.

Proof For ease of notations, we assume that $\sinh(\theta) \ne 0$. The case $\sinh(\theta) = 0$ can be treated by replacing $\sinh(i\theta)$ by $s_i(\theta)$ in the proof. First, we note that any real number λ can be written as $2\sqrt{k-1}\cosh(\theta)$, where θ is a complex number. Such λ is an eigenvalue of $M_{l+1}(k; \rho_0, 1, \ldots, 1)$ with eigenvector $(r_0, \ldots, r_l)$ iff

$$\lambda r_0 = k r_1 \quad (3)$$

$$\lambda r_1 = \rho_0 r_0 + (k-\rho_0) r_2 \quad (4)$$

$$\lambda r_i = r_{i-1} + (k-1) r_{i+1} \text{ for } 2 \le i \le l-1 \quad (5)$$

$$\lambda r_l = r_{l-1} \quad (6)$$

From Eqs. 5 and 6, we see that, up to a constant factor, $r_i = (k-1)^{\frac{-i}{2}}\sinh((l+1-i)\theta)$ for $i \ge 1$. Eqs. 3 and 4 imply that $\lambda^2 r_1 = \rho_0 k r_1 + \lambda(k-\rho_0)r_2$, which reduces to the equation in the lemma.

Claim 3 *If* θ *is a nonnegative real and* $l \ge 1$, *then*

$$\frac{l-1}{l}e^{-\theta} \le \frac{\sinh((l-1)\theta)}{\sinh(l\theta)} \le e^{-\theta} \quad (7)$$

Proof To prove inequality 7, we observe that

$$e^{-\theta} - \frac{\sinh((l-1)\theta)}{\sinh(l\theta)} = \frac{e^{-(l-1)\theta} - e^{-(l+1)\theta}}{e^{l\theta} - e^{-l\theta}} = e^{-l\theta}\frac{\sinh(\theta)}{\sinh(l\theta)} \le \frac{e^{-\theta}}{l},$$

since $\sinh(l\theta) \ge l\sinh(\theta)$. ∎

By setting $h(\theta') = (\lambda'^2 - k\rho_0)\sinh(l\theta') - 2(k-\rho_0)\cosh\theta'\sinh((l-1)\theta')$, where $\lambda' = 2\sqrt{k-1}\cosh\theta'$, we see that $h(\theta')$ goes to $+\infty$ as θ' goes to $+\infty$, and so the condition in the lemma implies that $h(\theta') \ge 0$. Using Eq. 7, we get

$$(\lambda'^2 - k\rho_0) \ge 2(k-\rho_0)\cosh(\theta')\frac{\sinh((l-1)\theta')}{\sinh(l\theta')} = 2(k-\rho_0)\cosh(\theta')e^{-\theta'}(1+O(\frac{1}{l}))$$

Therefore,

$$\rho_0(k - 2e^{-\theta'}\cosh\theta') \le (\lambda'^2 - 2ke^{-\theta'}\cosh\theta')(1+O(\frac{1}{l})) \quad (8)$$

since $e^{-\theta'}\cosh\theta' \le 1$, $k \ge 3e^{-\theta'}\cosh\theta'$ and $\lambda'^2 \ge 8k/3$. By factoring $2\cosh\theta'$ in the righthand side of Eq. 8 and noting that

$$(2k-2)\cosh\theta' - ke^{-\theta'} = k(2\cosh\theta' - e^{-\theta'}) - 2\cosh\theta' = ke^{\theta'} - 2\cosh\theta',$$

we see that Eq. 8 implies $\rho_0 \le 2\cosh(\theta')e^{\theta'}(1+O(\frac{1}{l}))$. ∎

Theorem 1 *If $G = (V, E)$ is k-regular and $\tilde{\lambda} = \max(\lambda_1(G), 2\sqrt{k-1})$, then for any non-empty subset X of size at most $k^{-1/\epsilon}|V|$,*

$$\frac{|N_G(X)|}{|X|} \geq \frac{k}{2}\left(1 - \sqrt{1 - \frac{4k-4}{\tilde{\lambda}^2}}\right)(1 + O(\epsilon)),$$

where the constant behind the O is a small absolute constant.

Proof Let $\tilde{\lambda} = 2\sqrt{k-1}\cosh(\theta)$, where $\theta \geq 0$. We apply lemma 4 with $t = 1$, $X_0 = X$ and $X_1 = N_G(X)$ and $l = \lfloor\frac{1}{2\epsilon}\rfloor$. Let $\lambda' = \tilde{\lambda} + 4k^l\frac{|X_t|}{n} = 2\sqrt{k-1}\cosh(\theta')$, where $\theta' \geq \theta \geq 0$. Since $|X_1| \leq k|X| \leq k^{1-\frac{1}{\epsilon}}n$, we have $|X_1|k^l \leq k^{1-\frac{1}{2\epsilon}}n$ and $\lambda' - \tilde{\lambda} \leq 4k^{1-\frac{1}{2\epsilon}} = O(\epsilon^2)$. This implies that $\lambda' = \tilde{\lambda}(1 + O(\epsilon))$ and $\cosh\theta' - \cosh\theta = O(\epsilon^2)$. Using the inequality $(x-y)^2 \leq 2(\cosh x - \cosh y)$ valid for $x \geq y \geq 0$, we see that $\theta' - \theta = O(\epsilon)$. Lemma 5 then implies

$$\rho_0 \leq (1 + e^{2\theta'})(1 + O(\epsilon)) = (1 + e^{2\theta})(1 + O(\epsilon)).$$

Hence

$$\frac{|N_G(X)|}{|X|} = \frac{k}{\rho_0} \geq \frac{k}{2e^\theta\cosh\theta}(1 + O(\epsilon)).$$

Using the equalities $e^{-\theta} = \cosh\theta - \sinh\theta = \cosh\theta - \sqrt{\cosh^2\theta - 1}$, we get

$$\frac{1}{e^\theta\cosh\theta} = 1 - \sqrt{1 - \frac{1}{\cosh^2\theta}} = 1 - \sqrt{1 - \frac{4k-4}{\tilde{\lambda}^2}}$$

This concludes the proof. ■

Corollary 1 *If G is k-regular on n vertices, then $\lambda_1(G) \geq 2\sqrt{k-1}(1 + O(\log_k^{-2} n))$.*

Proof We apply lemma 4 with X_0 consisting of a single vertex, $t = 0$ and $l = \lfloor\frac{\log_k n}{2}\rfloor$. From the theory of non-negative matrices [20], we know that the largest eigenvalue of the matrix $M_l(k; 1, \ldots, 1)$ is no smaller than the largest eigenvalue of the same matrix where the $(1,2)$ entry is replaced by $k-1$. But a calculation similar to the one in lemma 5 shows that the largest eigenvalue of this matrix is $\cos(\frac{\pi}{l+1})$. Hence $\lambda_1(G) + 4\frac{k^l}{n} \geq 2\sqrt{k-1}\cos(\frac{\pi}{l+1})$. We conclude the proof by noting that $\frac{k^l}{n} = O(\log_k^{-2} n))$ and $\cos(\frac{\pi}{l+1}) = 1 + O(\log_k^{-2} n)$. ■

As we mentionned in the introduction, the lower bound in Corollary 1 was independently obtained in [8].

4 A family of "almost" Ramanujan graphs with expansion $k/2$

Lemma 6 *Given $k \geq 3$, a real $\theta > 0$ and $\lambda = 2\sqrt{k-1}\cosh\theta$, let $S_l(a,b)$ be the sequence defined for $l \geq 0$ by $S_0(a,b) = a$, $S_1(a,b) = b$, $\lambda S_1(a,b) = 2S_0(a,b) + (k-2)S_2(a,b)$ and the recurrence relation*

$$\lambda S_l(a,b) = S_{l-1}(a,b) + (k-1)S_{l+1}(a,b), \text{ for } l \geq 2, \tag{9}$$

where a and b are real numbers. If $b_1 + \ldots + b_k = \lambda a$, then $\forall l \geq 1$,

$$\sum_{i=1}^k S_l{}^2(a, b_i) \geq (k-1)^{1-l}\cosh^2((l-1)\theta)\sum_{i=1}^k b_i{}^2$$

Proof First we show that, for $l \geq 0$,

$$S_{l+1}(k,\lambda) = 2(k-1)^{\frac{1-l}{2}}(\cosh\theta\cosh l\theta + \tfrac{k}{k-2}\sinh\theta\sinh l\theta) \tag{10}$$

We first notice that the two sequences $(k-1)^{-l/2}\cosh(l\theta)$ and $(k-1)^{-l/2}\sinh(l\theta)$, form a basis to the set of sequences satisfying the recurrence 9. Therefore, to prove Eq. 10, it suffices to show that the two sides are equal for $l = 0$ and $l = 1$. The case $l = 0$ is straightforward. For $l = 1$, we have

$$\begin{aligned} S_2(k,\lambda) &= \frac{\lambda S_1(k,\lambda) - 2S_0(k,\lambda)}{k-2} \\ &= \frac{\lambda^2 - 2k}{k-2} \\ &= \frac{4(k-1)\cosh^2\theta - 2k(\cosh^2\theta - \sinh^2\theta)}{k-2}, \end{aligned}$$

which is equal to the righthand side of Eq. 10.

Since $S_l(a,b)$ is a linear function of (a,b), we have $S_l(a,b) = c_l a + d_l b$, where $c_l = S_l(1,0)$ and $d_l = S_l(0,1)$. Therefore,

$$\begin{aligned} \sum_{i=1}^k S_l{}^2(a,b_i) &= \sum_{i=1}^k (c_l{}^2a^2 + 2c_ld_lab_i + d_l{}^2b_i^2) \\ &= (kc_l + 2\lambda d_l)c_la^2 + d_l{}^2\sum_{i=1}^k b_i^2 \end{aligned}$$

But, using the same reasoning as before, we know that $c_l = (k-1)^{1-\frac{l}{2}}c_2\frac{\sinh(l-1)\theta}{\sinh\theta} \leq 0$ since $c_2 = \frac{-2}{k-2}$ and so $kc_l + 2\lambda d_l = 2S_l(k,\lambda) - kc_l \geq 0$. On the other hand, $k\sum_{i=1}^k b_i^2 \geq \lambda^2a^2$ by Cauchy-Schwarz. Hence,

$$\sum_{i=1}^k S_l{}^2(a,b_i) \geq ((kc_l + 2\lambda d_l)c_l\frac{k}{\lambda^2} + d_l{}^2)\sum_{i=1}^k b_i^2$$

But $(kc_l + 2\lambda d_l)c_l\frac{k}{\lambda^2} + d_l{}^2 = (\frac{k}{\lambda}c_l + d_l)^2 = \frac{S_l{}^2(k,\lambda)}{\lambda^2}$. We conclude by noting that $S_l(k,\lambda) \geq 2(k-1)^{1-\frac{l}{2}}\cosh\theta\cosh(l-1)\theta$. ∎

Lemma 7 *If $G=(V,E)$ is k-regular on n vertices, for any $f \in L^2(V)$, we have*

$$f\cdot Af \leq \lambda_1(G)||f||^2 + \frac{k}{n}\Big(\sum_{v\in V} f(v)\Big)^2$$

Proof Let $\bar{f} = \frac{f\cdot\chi_V}{n}\chi_V$ be the orthogonal projection of f on the space spanned by the constant vector χ_V. Then $f_0 = f - \bar{f}$ is the orthogonal projection of f on $L_0^2(V)$. We have $Af = A\bar{f} + Af_0 = k\bar{f} + Af_0$, and so $f\cdot Af = k||\bar{f}||^2 + f_0\cdot Af_0$ since $\bar{f}\cdot Af_0 = A\bar{f}\cdot f_0 = k\bar{f}\cdot f_0 = 0$. But $||\bar{f}||^2 = \frac{(f\cdot\chi_V)^2}{n^2}||\chi_V||^2 = \left(\sum_{v\in V} f(v)\right)^2/n$, and $f_0\cdot Af_0 \leq \lambda_1(G)||f_0||^2 \leq \lambda_1(G)||f||^2$ since $||f||^2 = ||f_0||^2 + ||\bar{f}||^2$. ∎

Theorem 2 *For any integer k such that $k-1$ is prime, we can explicitly construct an infinite family of k-regular graphs G_n on n vertices whose linear expansion is $\frac{k}{2}$ and such that $\lambda_1(G_n) \leq 2\sqrt{k-1}(1+2\frac{\log^2\log n}{\log_k^2 n})$.*

Proof From [15] and [17], we know that we can explicitly construct an infinite family of bipartite Ramanujan graphs H_n on n vertices whose girth is at least $(4/3+o(1))\log_{k-1} n$. Let $H_n = (V,E)$ be an element of the family, $u \in V$ a vertex of H_n and $t = \lfloor\frac{c(H_n)}{2}\rfloor - 2$. For $0 \leq l \leq t+1$, let V_l denote the set of nodes in H_n at distance l from v. Let $u_1,\ldots,u_k$ be the neighbors of u and let $v_1,\ldots,v_k$ be k vertices of V_2 such that $(u_i,v_i)\in E$. The subgraph of H_n induced on the subset $\bigcup_{l=0}^{t+1} V_l$ is a tree rooted at u since it is connected and contains no cycle. We will denote by T_i the subtree rooted at u_i minus the subtree rooted at v_i. Let u' and v' be two elements not belonging to V. Consider the k-regular graph $G_{n+2} = (V',E')$, where $V' = V\cup\{u',v'\}$ and $E' = E\cup\bigcup_{i=1}^k\{(u',u_i),(u_i,u'),(v',v_i),(v_i,v')\} - \bigcup_{i=1}^k\{(u_i,v_i),(v_i,u_i)\}$. Figure 1 shows the graph G_{n+2} in the neighborhood of u in the case $k=3$. For shorthand, we denote $A(G_{n+2})$ by A' and $\lambda_1(A')$ by λ_1. Assume that $\lambda_1 > 2\sqrt{k-1}$ (otherwise we are done), and let $\lambda_1 = 2\sqrt{k-1}\cosh\theta$, with $\theta > 0$. Let $g \in L_0^2(V')$ be an eigenvector corresponding to λ_1. Since u and u' have the same neighbors in G_{n+2} and

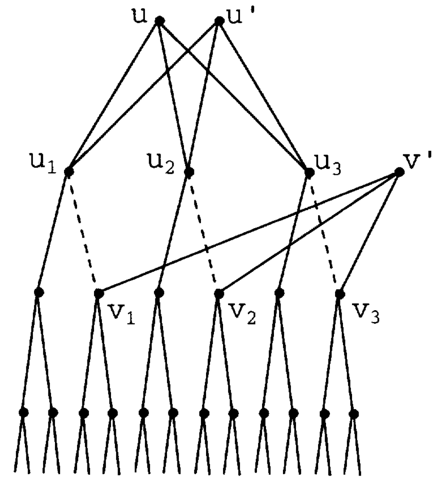

Figure 1: The graph G_{n+2} in the neighborhood of u in the case $k=3$. The dotted edges are those belonging to $E-E'$.

$\lambda_1 \neq 0$, we have $g(u) = g(u')$. Let f be the element of $L^2(V)$ that coincides with g on V. By Eq. 2, we have

$$\begin{aligned}\lambda_1||g||^2 &= g\cdot A'g\\ &= f\cdot A(H_n)f - 2\sum_{i=1}^k g(u_i)g(v_i) +\\ &\quad 2\sum_{i=1}^k g(u')g(u_i) + 2\sum_{i=1}^k g(v')g(v_i)\\ &\leq f\cdot A(H_n)f + \sum_{i=1}^k (g(u_i)^2 + g(v_i)^2) +\\ &\quad 2\lambda_1 g(u')^2 + 2\lambda_1 g(v')^2 \qquad (11)\end{aligned}$$

In the third inequality, we used the equation $(A'g)(u') = \lambda_1 g(u')$ and $(A'g)(v') = \lambda_1 g(v')$. Note that $\sum_{w\in V} f(w) = -g(u') - g(v')$ since $g \in L_0^2(V')$. Using lemma 7, we get

$$\begin{aligned}f\cdot A(H_n)f &\leq \lambda_1(H_n)||f||^2 + \frac{k}{n}(g(u')+g(v'))^2\\ &\leq 2\sqrt{k-1}(||g||^2 - g(u')^2 - g(v')^2) +\\ &\quad \frac{2k}{n}(g(u')^2 + g(v')^2)\\ &\leq 2\sqrt{k-1}||g||^2,\end{aligned}$$

for sufficiently large n. Combining this with Eq. 11, we obtain

$$\begin{aligned}\lambda_1\|g\|^2 \le& 2\sqrt{k-1}\|g\|^2 + \sum_{i=1}^{k}(g(u_i)^2 + g(v_i)^2) + \\ & 2\lambda_1 g(u')^2 + 2\lambda_1 g(v')^2 \\ \le& 2\sqrt{k-1}(\|g\|^2 + 2\sum_{i=1}^{k}(g(u_i)^2 + g(v_i)^2)) \quad (12)\end{aligned}$$

The second inequality follows from

$$g(u')^2 = \frac{1}{\lambda_1^2}\Big(\sum_{i=1}^{k} g(u_i)\Big)^2 \le \frac{k}{\lambda_1^2}\sum_{i=1}^{k} g(u_i)^2$$

and a similar relation corresponding to v'. For $1 \le i \le k$ and $1 \le l \le t+1$, let $\mu_{i,0} = g(u)$ and $\mu_{i,l} = \sum_{w\in T_i\cap V_l} g(w)/|T_i \cap V_l|$. For any $w \in T_i \cap V_l$, with $l \ge 2$, we know that $\lambda_1 g(w)$ is equal to the sum of the values of g on the parent of w and on the $k-1$ children of w in T_i. By summing up these equalities, we see that $\lambda_1\mu_{i,l} = \mu_{i,l-1} + (k-1)\mu_{i,l+1}$. Similarly, $\lambda_1\mu_{i,1} = 2\mu_{i,0} + (k-2)\mu_{i,2}$. Hence $\mu_{i,l} = S_l(g(u), g(u_i))$, for $1 \le l \le t+1$, where S_l is the function defined in lemma 6. Therefore, using Cauchy-Schwarz,

$$\begin{aligned}\|g\|^2 &\ge \sum_{i=1}^{k}\sum_{w\in T_i\cap V_{t+1}} g(w)^2 \\ &\ge (k-2)(k-1)^{t-1}\sum_{i=1}^{k} S_{t+1}{}^2(g(u), g(u_i)) \\ &\ge \frac{1}{2}\cosh^2(t\theta)\sum_{i=1}^{k} g(u_i)^2.\end{aligned}$$

Similarly, we can prove the inequality $\|g\|^2 \ge \frac{1}{2}\cosh^2(t\theta)\sum_{i=1}^{k} g(v_i)^2$ by using a slightly modified version of lemma 6. Combining this with Eq. 12 yields

$$\cosh\theta \le 1 + \frac{8}{\cosh^2 t\theta} \quad (13)$$

Using the inequalities $\cosh\theta \ge 1 + \frac{\theta^2}{2}$ and $\cosh t\theta \ge e^{t\theta}/2$, we see from Eq. 13 that $\theta = O(e^{-t\theta})$ and $\log\theta \le -t\theta + O(1)$. Hence $\theta \le \frac{\log t}{t}$ for sufficiently large n. Finally, $\lambda_1 = 2\sqrt{k-1}\cosh\theta = 2\sqrt{k-1}(1 + \frac{\theta^2}{2}(1+o(1))) \le 2\sqrt{k-1}(1 + 2\frac{\log^2\log n}{\log_k^2 n})$ since $t \ge (\frac{2}{3} + o(1))\log_k n$. Theorem 1 implies that the linear expansion of the family G_n is at least $\frac{k}{2}$. Since the subset $\{u, u'\}$ has k neighbors, this bound is tight. ■

If $k-1$ is a prime congruent to 1 modulo 4, we know from [15] that there exists an infinite family of non-bipartite k-regular Ramanujan graphs with girth at least $\frac{2}{3}(1+o(1))\log_{k-1} n$. By doing the same construction as in Theorem 2, we obtain k-regular graphs whose second largest eigenvalue *in absolute value* is $2\sqrt{k-1}(1+o(1))$ and linear expansion $k/2$. Moreover, by adding nodes at regions of the graph at sufficiently large distance from each other, we can construct for any $m = m(n) = o(n)$ a family of k-regular graphs whose second largest eigenvalue in absolute value is $2\sqrt{k-1}(1+o(1))$ and containing a subset of size $2m$ with expansion $k/2$. The proof of these two statements is very similar to the proof of Theorem 2.

5 Examples and applications

1. **Number of edges in an induced subgraph.** Let $G = (V, E)$ be a k-regular graph on n vertices and Z_0 a proper subset of V. We apply lemma 1 and the remark following it with $t = 1$ and $Z_1 = V - Z_0$. Note that $\lambda_0(G) = \lambda_0(\Phi) = k$ since $\sum_{j=0}^{t} e(Z_i, Z_j) = e(Z_i, V) = k|Z_i|$. Therefore, $k + \lambda_1(\Phi) = \text{trace}(\Phi) = \frac{e(Z_0,Z_0)}{|Z_0|} + \frac{e(Z_1,Z_1)}{|Z_1|}$. But $e(Z_1, Z_1) = k|Z_1| - e(Z_1, Z_0) = k|Z_1| - k|Z_0| + e(Z_0, Z_0)$, and so $|Z_0||Z_1|(k + \lambda_1(\Phi)) = ne(Z_0, Z_0) + k|Z_1||Z_0| - k|Z_0|^2$. The inequality $|\lambda_1(\Phi)| \le \lambda$ then becomes $|e(Z_0, Z_0) - k\frac{|Z_0|^2}{n}| \le \lambda|Z_0|(1 - \frac{|Z_0|}{n})$. Note that this inequality is also true in the case $Z_0 = \emptyset$ or $Z_0 = V$. This result has already been established in [4].

2. **Tanner's inequality.** Again, we assume that $G = (V, E)$ is a k-regular graph on n vertices. Let X be a proper subset of V. We apply lemma 2 with $t = 3$, $X_0 = X$, $X_1 = N_G(X)$ and $X_2 = X_3 = V$. We have $\rho_0 = k\frac{|X|}{|X_1|}$, $\rho_1 = k\frac{|X_1|-|X|}{|V|-|X|}$ and $\rho_2 = k$. As before, k is an eigenvalue of $M_4(k; \rho_0, \rho_1, k)$. Since the (i,j) entry of this matrix is null if $i \equiv j \pmod 2$, the other eigenvalues of $M_4(k; \rho_0, \rho_1, k)$ are $\{\sigma, -\sigma, -k\}$, with $|\sigma| \le \lambda(G)$. But $\det(M_4(k; \rho_0, \rho_1, k)) = k^2\rho_0(k - \rho_1) = k^2\sigma^2$, and so $\rho_0(k - \rho_1) \le \lambda(G)^2$. A simple calculation shows that this implies Eq. 1. Note that we implicitly assumed that $N_G(X)$ is a proper subset of V. However, we can directly check that Eq. 1 holds if $N_G(X) = V$.

3. **Random regular graphs.** In [7], it was shown that, if k is even, then for "most" k-regular graphs G, we have $\lambda_1(G) \le 2\sqrt{k-1} + O(\log k)$. Therefore, using Theorem 1, we see that for "most" regular graphs, we can prove *in polynomial time* that linear sized subsets have expan-

sion at least

$$\frac{k}{2}(1-\sqrt{1-\frac{4k-4}{(2\sqrt{k-1}+O(\log k))^2}}),$$

which is equal to $\frac{k}{2} - O(k^{3/4}\log^{1/2} k)$.

4. **Selection networks.** We can use Theorem 1 to build explicit selection networks of small size. A selection network is a network of comparators that classifies a set of n numbers, where n is even, into two subsets of $n/2$ numbers such that any element in the first set is smaller than any element in the second set. In [19], a probabilistic construction of a selection network is given using an asymptotic upper bound of $2n\log_2 n$ comparators. Also, an upper bound slightly less than $6n\log_2 n$ is shown by a deterministic construction. Using Theorem 1, we can [13] construct selection networks of asymptotic size $(3+\epsilon)n\log_2 n$, for any $\epsilon > 0$.

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