CS395T: Pseudorandomness and Combinatorial Constructions
Lecture 14, March 1, 2001
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1 Introduction

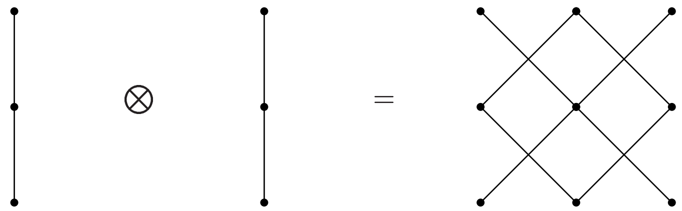
A *graph product* is a binary operator which creates a new graph from two input graphs. The resulting graph usually inherits some of the properties of the graphs from which it was created. A graph product provides one way of constructing complicated graphs out of simple components. First, we shall examine several canonical graph products. Then, we shall describe another operator, the zig-zag product, which allows us to build constant-degree expander graphs.

2 Canonical Graph Products

Let $G = (V, E)$ be a graph with transition matrix A and second eigenvalue λ. (Recall that $\lambda = \max\{|\lambda_2|, -\lambda_n\}$.) The *kth power* of G, which we denote by G^k, corresponds to k steps of a random walk on G. Therefore, the transition matrix of G^k is simply A^k. Since A is symmetric, the second eigenvalue of A^k equals λ^k.

If G is a small constant-degree expander, we might hope to obtain a large constant-degree expander. The product G^k is not useful here, since the new graph is the same size as the old graph. Moreover, the degree increases with k. Let us examine a different type of product which will yield a larger graph.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The *tensor product* of G_1 and G_2, which we denote by $G = G_1 \otimes G_2$, is a new graph on the set of vertices $V = V_1 \times V_2$. The graph G contains the edge $\{(v_1, v_2), (w_1, w_2)\}$ if and only if $\{v_1, w_1\} \in E_1$ and $\{v_2, w_2\} \in E_2$. Some examples may help clarify the definition. The tensor product of two lines is a diagonal lattice:



By wrapping this diagram around, we see that the tensor product of two cycles is the disjoint union of two tori: $C^n \otimes C^n = \mathbb{T} \cup \mathbb{T}$.

Now, we introduce a notation which indicates some of the characteristics of regular graphs. We say G is (n, d, λ), where n is the size of the vertex set, d is the degree of the graph and λ is the second eigenvalue.

Let G_1 be (n_1, d_1, λ) and G_2 be (n_2, d_2, μ). Then the tensor product $G = G_1 \otimes G_2 = (n_1 n_2, d_1 d_2, \max\{\lambda, \mu\})$. It is clear that the size of the vertex set and the degree of the new graph are given by these formulae. It is more complicated to see that the the second eigenvalue of the tensor product equals the maximum of the second eigenvalues of the factors. The reason this relation holds is that the transition matrix is given by the tensor product of the transition matrices of its factors, $A = A_1 \otimes A_2$. The eigenvalues of A equal the products $\alpha_i \beta_j$, where $\{\alpha_i\}$ are the eigenvalues of A_1 and $\{\beta_j\}$ are the eigenvalues of A_2. Since $\max\{\alpha_i\} = 1$ and $\max\{\beta_j\} = 1$, the second largest value of $\alpha_i \beta_j$ must equal $\max\{\lambda, \mu\}$.

We have moved closer to our goal of producing a large expander, since the tensor product graph is somewhat larger than its factors and the second eigenvalue remains fairly small. The degree, however, increases unacceptably.

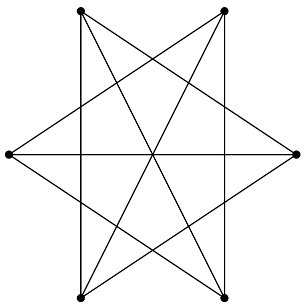
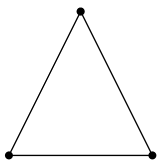
3 The Zig-Zag Product

Recently, Reingold, Vadhan and Wigderson have developed another type of graph product which is useful for building constant-degree expanders. Let G be (n_1, d_1, α) and H be (d_1, d_2, β) graphs. We shall define the zig-zag product $G \textcircled{z} H$ which is $(n_1 d_1, d_2^2, \alpha + \beta + \beta^2)$.

To see how this is useful, let us consider a specific example. Choose H to be $(d^4, d, 1/5)$ and put G_1 as $(d^4, d^2, 2/5)$. Now, we define a sequence $G_{i+1} = G_i^2 \textcircled{z} H$. If G_i is $(n_i, d^2, 2/5)$, then G_{i+1} is $(n_i d^4, d^2, 2/5)$. Using induction, we see that G_k will be $(d^{4k}, d^2, 2/5)$. Therefore, we can easily obtain a very large constant-degree expander with a small second eigenvalue. As we saw in the previous lecture, G_k is suitable for amplification the success probability of an RP algorithm.

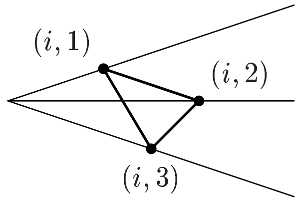
An even more efficient construction of expanders from zig-zag products uses the same graph G_1 as before along with a new second graph H which is $(d^8, d, 1/5)$. The recursion sets $G_{i+1} = (G_i \otimes G_i)^2 \textcircled{z} H$.

The easiest way to understand the construction of a zig-zag product is probably by example. To wit, let

$G =$  and $H =$  .

So, we see that $G = (6, 3, 2/3)$ and $H = (3, 2, 1/2)$. We define a new graph $Z = G \textcircled{z} H$ as follows.

For each of the n_1 vertices of G, the zig-zag product has a *cloud* of d_1 associated vertices. Intuitively, we imagine imbedding a copy of H at each vertex of G. We denote the vertices of Z by the symbol (i, j), where i indicates the associated vertex of G and j indicates the associated vertex of H. In our example, each vertex i of G resembles the following diagram:



Each vertex (i, j) of Z has d_2^2 neighbors, which we determine using the following procedure:

(Zig) 1. Choose a neighbor j^* of j in the graph H, for which there are d_2 options.

2. Let i' be the neighbor of i in the graph G which we obtain by following edge number j^*.

(Zag) 3. Choose a neighbor j' of j^* in the graph H, for which there are d_2 options.

4. We say that the vertex (i', j') is a neighbor of the vertex (i, j) in the graph Z.

Because the numbering of edges from each vertex of G is arbitrary, we must adopt a numbering convention, or else we must show that $G \circledz H$ is well-defined. But we shall not discuss this point further.

This zig-zag procedure is difficult to illustrate with a simple diagram. The essential idea is that a random walk on the zig-zag product involves taking a random step in the starting cloud and sliding across the associated edge to a second cloud before taking another random step in that second cloud. We thereby inherit a small mixing time from the factor graphs.

Now, we present a partial proof that the second eigenvalue of the zig-zag product is bounded above by $\alpha + \beta + \beta^2$. Let A, B be the transition matrices of G and H, and let M be the transition matrix of $G \circledz H$. We seek to bound the Rayleigh quotient

$$\rho(\mathbf{x}) = \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

for $\mathbf{x} \in \mathbb{R}^{n_1 d_1}$ which is orthogonal to $\mathbf{1}$. Intuitively, we view $\mathbf{x}$ as the nonuniform component of the probability distribution on the zig-zag product. In effect, we are showing that the distribution becomes more uniform by a factor of $\alpha + \beta + \beta^2$ after each step of a random walk.

To make the analysis, we must represent the overall probability distribution $\mathbf{x}$ in terms of the probability distributions on each cloud, which we denote $\mathbf{x}^j \in \mathbb{R}^{d_1}$ and define as $(\mathbf{x}^j)_i = \mathbf{x}_{ij}$, where the subscript indicates the product of the indices i and j. We may write this relationship more compactly as

$$\mathbf{x} = \sum_j e_j \otimes \mathbf{x}^j,$$

where e_j are the canonical basis vectors for $\mathbb{R}^{n_1}$. We may decompose each of the vectors $\mathbf{x}^j$ uniquely into components $(\mathbf{x}^j)^{\|}$ parallel with $\mathbf{1} \in \mathbb{R}^{d_1}$ and $(\mathbf{x}^j)^{\perp}$ orthogonal to $\mathbf{1}$. So we write

$$\begin{aligned}
\mathbf{x} &= \sum_j e_j \otimes ((\mathbf{x}^j)^{\|} + (\mathbf{x}^j)^{\perp}) \\
&= \sum_j e_j \otimes (\mathbf{x}^j)^{\|} + \sum_j e_j \otimes (\mathbf{x}^j)^{\perp} \\
&= \mathbf{x}^{\|} + \mathbf{x}^{\perp}
\end{aligned}$$

which we take as the definition of $\mathbf{x}^{\|}$ and $\mathbf{x}^{\perp}$.

We may also decompose the zig-zag transition matrix into a product $M = \tilde{B}\tilde{A}\tilde{B}$, which represents the three steps necessary to determine adjacency in the zig-zag graph. We can write $\tilde{B} = I \otimes B$, where I is the identity on $\mathbb{R}^{n_1}$. The definition of the permutation matrix $\tilde{A}$ is more subtle, so we refer the interested reader to Reingold et al. Since $\tilde{B}$ is symmetric, we see that

$$\mathbf{x}^T M \mathbf{x} = \mathbf{x}^T \tilde{B}\tilde{A}\tilde{B}\mathbf{x} = (\tilde{B}\mathbf{x})^T \tilde{A}\tilde{B}\mathbf{x}.$$

We also observe that $\tilde{B}\mathbf{x}^{\|} = \mathbf{x}^{\|}$ because $\mathbf{1}$ is a 1-eigenvector of B. We obtain

$$\mathbf{x}^T M \mathbf{x} = (\mathbf{x}^{\|} + \tilde{B}\mathbf{x}^{\perp})^T \tilde{A}(\mathbf{x}^{\|} + \tilde{B}\mathbf{x}^{\perp}).$$

Expanding this expression and using the fact that $\tilde{A}$ preserves distances (because it is a permutation matrix), we reach the estimate

$$|\mathbf{x}^T M\mathbf{x}| \leq |(\mathbf{x}^{||})^T \tilde{A}\mathbf{x}^{||}| + 2||\mathbf{x}^{||}||\,||\tilde{B}\mathbf{x}^{\perp}|| + ||\tilde{B}\mathbf{x}^{\perp}||^2.$$

We make the claim that $||\tilde{B}\mathbf{x}^{\perp}|| \leq \beta||\mathbf{x}^{\perp}||$. We also claim that $|(\mathbf{x}^{||})^T\tilde{A}\mathbf{x}^{||}| \leq \alpha||\mathbf{x}^{||}||^2$. For proof, refer to Reingold et al. Substituting these relations into our last estimate, we have

$$|\mathbf{x}^T M\mathbf{x}| \leq \alpha||\mathbf{x}^{||}||^2 + 2\beta||\mathbf{x}^{||}||\,||\mathbf{x}^{\perp}|| + \beta^2||\mathbf{x}^{\perp}||^2.$$

Finally, we put $p = ||\mathbf{x}^{||}||/||\mathbf{x}||$ and $q = ||\mathbf{x}^{\perp}||/||\mathbf{x}||$, and note that $p^2 + q^2 = 1$. We conclude that the Rayleigh quotient has the bound

$$\rho(\mathbf{x}) = \frac{|\mathbf{x}^T M\mathbf{x}|}{\mathbf{x}^T\mathbf{x}} \leq \alpha p^2 + 2\beta pq + \beta^2 q^2 \leq \alpha + \beta + \beta^2.$$