On the Time to Traverse All Edges of a Graph

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Abstract

The expected time for a random walk on an undirected graph $G = (V, E)$ to visit all the vertices is $O(|V||E|)$ [AKLLR], and is $O(|V|^2)$ for regular graphs [KLNS]. Here we show that both bounds hold even if we are required to traverse all the edges, although our bound for regular graphs requires that the degree be $|V|^\delta$ for some $\delta < 1$.

Keywords: random walk, cover time, combinatorial problems

1 Introduction

A random walk on an undirected graph is the sequence of vertices visited by a particle that starts at a specified vertex and visits other vertices according to the following transition rule: if the particle is at vertex i at time t, then at time $t + 1$ it moves to a neighbor of i picked uniformly at random.

Simulating a random walk on a graph requires very local information about the graph, while random walks have very nice global properties. This makes random walks very useful in computation, where limited resources are available to determine global information. Bounds on the time of random walks to visit all vertices were important in showing that UNDIRECTED *st*-CONNECTIVITY can be computed in RSPACE($\log n$) [AKLLR] and in analyzing the simulation of token rings on arbitrary networks [BK].

Aleliunas et. al. [AKLLR] showed that the expected time to visit all the vertices, called the cover time, is $O(|V||E|)$. A natural extension is the expected time to traverse all the directed edges, where a random walk traverses directed edge (v, w) if it visits v and w consecutively. The most natural method gives a bound of $O(|V||E| \log |V|)$. In this note we obtain a bound of $O(|V||E|)$ for this problem.

For regular graphs, Kahn et. al. [KLNS] have improved the bound on visiting all vertices to $O(|V|^2)$. Here again the most natural way to prove a similar result about traversing all directed edges adds a $\log |V|$ factor, giving $O(|V|^2 \log |V|)$. We obtain a bound of $O(|V|^2)$ if the degree is $|V|^\delta$ for some $\delta < 1$.

We prove all of our bounds in the more general setting of a directed graph where the indegree equals the outdegree at every node. Our techniques rely on known bounds for c_v.

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2 Notation

The random walk is performed on $G(V, E)$, a directed graph with indegree equal to outdegree everywhere. We define the following:

$n = |V|$.

$m = |E|$.

$E_x(\cdot)$ is the expected value of $(\cdot)$ in a random walk starting at vertex or edge x.

T_x is the time to first visit vertex or edge x.

$t_{max} = \max_{u,v \in V}\{E_u T_v\}$.

C_v is the time to first visit all the vertices.

$c_v = \max_{u \in V}\{E_u C_v\}$.

C_e is the time to first traverse all the directed edges.

$c_e = \max_{u \in V}\{E_u C_e\}$.

3 Main Result

We use the following lemma repeatedly:

Lemma 1 *For any edge (v, w), $E_v T_{v,w} < m$.*

Proof. $E_v T_{v,w} < E_w T_{v,w} = E_{v,w} T_{v,w} = m$, where this last equality is a standard fact (see e.g. [AKLLR]). ■

To get the most obvious upper bound on c_e, we note that for any two edges e_1 and e_2, $E_{e_1} T_{e_2} \leq t_{max} + m$. Thus the results of e.g. [Ma] imply that $c_e = O(t_{max} \log m + m \log m)$. For those graphs having $c_v = \Omega(t_{max} \log n)$, such as the hypercube and k-dimensional toruses for $k > 1$ (see [A1] and [Z]), this is tight to within a constant factor. This is because $c_e \geq c_v$, $c_v = O(t_{max} \log n)$ (see e.g. [Ma]), and that $c_e = \Omega(m \log m)$ (see [A2]).

For general graphs, however, it gives a bound of $c_e = O(mn \log n)$. To improve this to $O(mn)$, we prove the following lemma:

Lemma 2 *For any positive integer k, $c_e = O(k(c_v + m^{1+\frac{1}{k}}))$.*

Proof. The idea we use is that once the walk reaches a vertex v, it is likely to quickly traverse all of the directed edges coming out of v. What takes longer is to travel between two distant vertices. Therefore, if we cover all the vertices and walk just a little longer, we will make reasonable progress in covering all the edges.

More precisely, let the random walk begin at x. We divide our walk into T shorter walks: the ith walk lasts for S_i steps, where S_i is the stopping time defined as follows: first cover all the vertices, then walk another $m^{1+\frac{1}{k}}$ steps; the time of the next visit to x is called S_i. Here T is the stopping time

$$T = \min\{t | \text{all edges are traversed after } S_1 + S_2 + \ldots + S_t \text{ steps}\}.$$

Then the S_i's are i.i.d., so using Wald's identity (see e.g. [D], p.156),

$$E_x C_e \leq E_x[\sum_{i=1}^{T} S_i] = (E_x S_i)(E_x T). \tag{1}$$

Moreover,

$$E_x S_i \leq c_v + m^{1+\frac{1}{k}} + \max_u \{E_u T_x\} \leq 2c_v + m^{1+\frac{1}{k}}. \tag{2}$$

Thus it suffices to bound $E_x T$. Now define $B_{i,u,w}$ as the event in which the ith walk (for S_i steps) fails to traverse the directed edge (u, w). Because $E_u T_{(u,w)} < m$ and we walk for at least $m^{1+\frac{1}{k}}$ steps after visiting u,

$$Pr[B_{i,u,w}] < m^{-1/k}.$$

Therefore,

$$Pr[T > 2k] = Pr[(\exists \text{edge } (u, w))(\forall i, 1 \leq i \leq k) B_{i,u,w}] < m(m^{-1/k})^{2k} = \frac{1}{m}.$$

Because each new walk beginning at x is independent of previous ones,

$$Pr[T > 2kj] < \frac{1}{m^j}.$$

Thus

$$E_x T < 2k \sum_{j=0}^{\infty} \frac{1}{m^j} = \frac{2k}{1 - 1/m},$$

and using (1) and (2) gives the lemma.

■

Taking $k = 2$, for example, gives the general $O(mn)$ upper bound.

For regular graphs with $m = n^{2-\delta}$, we can take $k > \frac{2-\delta}{\delta}$ in the above lemma to achieve an $O(n^2)$ bound. We remark that this bound cannot hold for all regular graphs, because the complete graph takes time $\Theta(n^2 \log n)$.

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