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A TECHNIQUE FOR LOWER BOUNDING THE COVER TIME*

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Abstract. A general technique for proving lower bounds on expected covering times of random walks on graphs in terms of expected hitting times between vertices is given. This technique is used to prove

(i) A tight bound of $\Omega(|V|\log^2|V|)$ for the two-dimensional torus;

(ii) A tight bound of $\Omega(|V|\log^2|V|/\log d_{\max})$ for trees with maximum degree $d_{\max}$;

(iii) Tight bounds of $\Omega(\mu^+\log|V|)$ for rapidly mixing walks on vertex transitive graphs, where μ^+ denotes the maximum expected hitting time between vertices.

In addition to these new results, the technique allows several known lower bounds on cover times to be systematically proved, often in a much simpler way.

Finally, a different technique is used to prove an $\Omega(1/(1-\lambda_2))$ lower bound on the cover time, where λ_2 is the second largest eigenvalue of the transition matrix. This was previously known only in the case where the walk starts in the stationary distribution [*J. Theoret. Probab.*, 2 (1989), pp. 101–120].

Key words. random walks, cover time, torus, trees, vertex transitive

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1. Introduction. A random walk on an undirected graph is the sequence of vertices visited by a particle that starts at a specified vertex and visits other vertices according to the following transition rule: if the particle is at vertex i at time t, then at time $t+1$ it moves to a neighbor of i picked uniformly at random. In this paper, we analyze the expected cover time, i.e., the expected time of the random walk to visit all the vertices.

Simulating a random walk on a graph requires very local information about the graph, while random walks have very nice global properties. This makes random walks useful in computation, where limited resources are available to determine global information. For example, random walks have proved useful in designing approximation algorithms for counting problems (see, e.g., [DFK] and [JS]), simulating complexity classes with few random bits [AKS], and assigning processes to nodes in networks [BC]. Bounds on cover times, in particular, were important in showing that UNDIRECTED *st*-CONNECTIVITY can be computed in RSPACE(log n) [AKLLR] and in analyzing the simulation of token rings on arbitrary networks [BK].

To understand what is known about cover times, consider for the moment the maximum expected cover time cov, where the maximum is taken over all start vertices. For interesting graphs, changing the start vertex changes the expected cover time by at most a constant factor, so analyzing cov is not really a restriction. Define E_iT_j to be the expected time to get from vertex i to vertex j, and $\mu^+ = \max_{i,j}\{E_iT_j\}$. It is not hard to show that μ^+ characterizes cov to within a log n factor, where $n = |V|$, i.e., that $\mu^+ \leqq \text{cov} \leqq O(\mu^+\log n)$ (see, e.g., [Z]). Good techniques have been developed to estimate μ^+, and involve calculating resistances of graphs [CRRST] and eigenvalues [A2]. Indeed, the E_iT_j's are computable in polynomial time, while it is not known if cov is. Therefore, the difficult part in establishing tight bounds for cov tends to be deciding the extra log n factor.

The basic technique for showing upper bounds of $O(\mu^+)$ is based on spanning trees, first used in [AKLLR] to show an upper bound for all graphs of $O(|V||E|)$, even though μ^+ can be $\Theta(|V||E|)$. In [KLNS] this technique is extended to show the general

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upper bound $O(|V||E|/d_{\min})$, and in [CRRST] another application of this technique is mentioned.

By contrast, a variety of techniques have been used to show the lower bound $\Omega(\mu^+ \log n)$, even though most of the work on lower bounds has been concentrated toward proving the conjectured lower bound of $\Omega(n \log n)$ for all graphs, regardless of the start vertex. Aldous [A3] has proved this bound if the walk starts from the stationary distribution. Examples of the different techniques are an inductive argument to show the $\Omega(n \log n)$ conjecture for trees [KLNS], a coupon-collector type argument to show the $\Omega(n \log n)$ conjecture for rapidly mixing walks [BK], and use of the $\Theta(\sqrt{n})$ standard deviation law to show an $\Omega(n \log^2 n/\log^2 d_{\max})$ lower bound for trees with small degree [Z].

In this paper, we present a general technique for showing the lower bound $\Omega(\mu^+ \log n)$ that yields all of the lower bounds described above except that given in [A3], as well as new lower bounds. All of our lower bounds are valid for any start vertex.

Our first bound is for the two-dimensional torus. The results in [A1] imply that the cover time for the k-dimensional torus is $\Theta(n \log n)$, for $k \geqq 3$. As the tight bound of $\Theta(n^2)$ is easy to show for the one-dimensional case, this only left open the time for the two-dimensional torus. It was known that $\mu^+ = \Theta(n \log n)$, which implied the best bounds on the cover time of $\Omega(n \log n)$ and $O(n \log^2 n)$ (see, e.g., [CRRST]). We show that the cover time is $\Theta(n \log^2 n)$.

Second, we improve the lower bound for trees in [Z] and [KLNS] to $\Omega(n \log^2 n/\log d_{\max})$; the case of the balanced k-ary tree shows that this is tight in terms of $d_{\max}$. This was obtained independently using a less general version of our method in [DS]. Aldous has since found the constant for balanced k-ary trees [A5].

Third, we give a lower bound for rapidly mixing walks. By rapidly mixing, we mean $\tau_2 \leqq n^{1-\delta}$, $\delta > 0$, where $\tau_2 = 1/(1-\lambda_2)$ is a measure of how quickly the random walk approaches stationarity (λ_2 is the second largest eigenvalue). This lower bound implies the $\Omega(n \log n)$ bound attained in [BK] for all rapidly mixing walks, as well as tight bounds of $\Theta(\mu^+ \log n)$ for rapidly mixing walks on vertex transitive graphs. This generalizes the result in [A1] showing this for Cayley graphs.

Finally, we use a different technique to show that the expected time to visit a vertex chosen at random according to the stationary distribution does not depend on the start vertex. This lemma implies the conjectured $\Omega(n \log n)$ lower bound for slowly mixing walks. Our $\Omega(\tau_2)$ lower bound was known previously only in the case where the walk starts from the stationary distribution [BK]. This leaves the $\Omega(n \log n)$ conjecture open only for the cases $n^{1-o(1)} \leqq \tau_2 \leqq n \log n$.

Our main technique is of interest in its own right, based on ideas in [Ma]. The difficult part in analyzing the cover time is correlations between hitting times of vertices; i.e., if the particle has visited i, what is the probability that it has visited j? We get around this by specifying random vertices to be visited; it is then easy to calculate the correlations between random vertices.

2. Notation. Let $G(V, E)$ be the graph on which the random walk is performed. For all of the following definitions, assume $i, j \in V$:

$n = |V|$,
d_i = degree of i,
$d_{\max} = \max_i \{d_i\}$,
$d_{\min} = \min_i \{d_i\}$,
$d(i,j)$ = distance between i and j.

Let $\{X_t\}$ be the sequence of vertices visited by the random walk, and let A be the associated transition matrix; i.e., $A_{ij} = 1/d_i$ if j is a neighbor of i, and 0 otherwise. Let

$1 = \lambda_1 > \lambda_2 \geqq \lambda_3 \geqq \cdots \geqq \lambda_n$ be the eigenvalues of A. Define

T_j = time to first reach j,
C = time to cover G,
$P_i(\cdot)$ denotes the probability of $(\cdot)$ in a walk starting at i,
$E_i(\cdot)$ denotes the expectation of $(\cdot)$ in a walk starting at i,
π is the stationary distribution, i.e., $\pi A = \pi$, so $\pi(i) = d_i/2|E|$,
$E_\pi(\cdot)$ denotes the expectation of $(\cdot)$ in a walk starting from distribution π,
$\mu^+ = \max_{i,j}\{E_iT_j\}$,
$\mu^- = \min_{i,j}\{E_iT_j\}$,
$H_k = 1 + 1/2 + \cdots + 1/k$ is the kth harmonic number.

3. The key lemma. Our lemma is based on the following theorem of Matthews [Ma].

THEOREM 1. *For any* $v \in V$, $\mu^- H_{n-1} \leqq E_vC \leqq \mu^+ H_{n-1}$.

We generalize the lower bound so that we still get an extra log n factor even if we allow, for each i, a polynomial fraction of the j to be close to i. We also allow i and j to be chosen from only a polynomial fraction of the vertices.

LEMMA 2. *Let* $V' \subseteq V$ *such that* $|V'| \geqq n^\alpha$, $\alpha > 0$, *and let* t *be such that for all* $i \in V'$, *at most* $1/n^\beta$ *fraction of the* $j \in V'$ *satisfy* $E_iT_j < t$, *where* $\beta > 0$. *Then for any* $v \in V$, $E_vC > t(\gamma \ln n - 2)$, *where* $\gamma = \min(\alpha, \beta)$.

Proof. We elaborate Matthews' idea by adding an extra element of randomness. Assume without loss of generality that the start vertex $v \in V'$. Let $y_1, \cdots, y_{|V'|-1}$ be a uniformly random permutation of $V' - \{v\}$. Let $Y_k = \{y_1, \cdots, y_k\}$. Let S_k be the first time that all the vertices in Y_k are visited, and let $R_k = S_k - S_{k-1}$. Note that $C \geqq S_{|V'|-1}$.

First, we claim that $P[R_k \neq 0] = 1/k$. The event $R_k \neq 0$ corresponds to y_k being visited after all of Y_{k-1}. We condition on a given walk occurring and on the set Y_k; the randomness left is in the order $y_1, \cdots, y_k$ of Y_k. Then y_k has probability $1/k$ of being the last element of Y_k visited in the walk. Since this is independent of what we condition on, the claim follows.

Now condition on the walk up to time S_{k-1} and on Y_{k-1}, and let $i = X_{S_{k-1}}$. Then, considering only the randomness involved in picking y_k, $P[E_iT_{y_k} < t] \leqq n^{\alpha-\beta}/(n^\alpha - k)$, by definition of t. Note that $E[R_k | R_k \neq 0] = E_iT_{y_k}$. Therefore, for $k \leqq n^\alpha/2$,

$$E[R_k] \geqq (P[R_k \neq 0] - P[E_iT_{y_k} < t])t \geqq \left(1/k - \frac{n^{\alpha-\beta}}{n^\alpha - k}\right)t \geqq (1/k - 2/n^\beta)t.$$

Thus,

$$E_vC \geqq ES_{|V'|-1} = \sum_{k=1}^{|V'|-1} ER_k \geqq \sum_{k=1}^{n^\gamma/2} ER_k$$

$$\geqq t\left(H_{n^\gamma/2} - \frac{n^\gamma}{2}\frac{2}{n^\beta}\right) \geqq t(\gamma \ln n - \ln 2 - 1) \qquad \square$$

4. Application to two-dimensional torus. The two-dimensional torus is the graph $G = (V, E)$, where $V = \{0, 1, 2, \cdots, r\}^2$ and a vertex (a, b) is connected to the four vertices $(a \pm 1 (\mathrm{mod}\ r), b)$ and $(a, b \pm 1 (\mathrm{mod}\ r))$.

It is known that $\mu^+ = \Theta(n \log n)$ (see, e.g., [CRRST]), which implies the only known bounds on the expected cover time of $\Omega(n \log n)$ and $O(n \log^2 n)$. We apply our key lemma to show a lower bound of $\Omega(n \log^2 n)$.

LEMMA 3. $E_iT_j = \Theta(n \log d(i, j))$. *In particular*, $E_iT_j > \frac{1}{4} n \ln 2d(i, j)$.

Proof. This follows easily from the ideas in [CRRST]. $\square$

Remark. Aldous [A6] has pointed out that using the results in [C] and some extra work, it is possible to replace the $\frac{1}{4}$ above by $1/\pi + o(1)$, hence improving the constant in Theorem 4 to $1/8\pi + o(1)$.

THEOREM 4. *For any* $v \in V$, $E_v C \geqq 1/32 n \ln n(\ln n - 3)$.

Proof. We apply the key lemma with $V' = V$ and $t = 1/16 n \ln n$. Note that there are at most $2d^2$ vertices j with $d(i, j) < d$. Taking $d = \frac{1}{2} n^{1/4}$, we see that there are $\leqq n^{1/2}/2$ vertices j with $d(i,j) < d$, and if $d(i,j) \geqq d$, then $E_i T_j > \frac{1}{4} n \ln n^{1/4} = t$. Thus, applying the key lemma with $\gamma = \beta = \log_n (n^{1/2}/2)$ gives the theorem. $\square$

5. Application to trees with small degree. Previous lower bounds for general trees have been $\Omega(n \log n)$ [KLNS] and $\Omega(n \log^2_{d_{\max}} n)$ [Z]. We improve both of these bounds to $\Omega(n \log n \log_{d_{\max}} n)$. This is the best possible, given only n and $d_{\max}$, as we show it is tight for balanced trees. To apply the key lemma, we must analyze $E_i T_j$ for trees, we need the following lemma from [Mo].

LEMMA 5. *For neighbors* i, j, $E_i T_j = 2|A_{ij}| - 1$, *where* A_{ij} *is the subtree containing* i *obtained by deleting edge* $\{i, j\}$.

COROLLARY 6. *In a tree*, $E_i T_j \geqq (d(i,j))^2$.

Proof. The above lemma implies that this time will be least in case our graph is a simple path from i to j, for which

$$E_i T_j = 1 + 3 + \cdots + 2d(i,j) - 1 = (d(i,j))^2. \qquad \square$$

LEMMA 7. *For any* i, *there are at most* $O(n^{3/4}\sqrt{\log n})$ *vertices* j *with* $E_i T_j < t = \frac{1}{4} n \log_{d_{\max}} n$.

Proof. We root our tree at i, and put parent-child relations on the vertices as usual. We construct a chain of vertices $i = i_1, i_2, \cdots, i_m$ as follows: we choose i_{j+1} to be the child of i_j with a subtree having at least $n/2$ vertices, if such exists, otherwise $m = j$.

Let I_j be the subtree with root i_j, so $I_j = A_{i_j, i_{j-1}}$ for $j > 0$, and set $I_{m+1} = \varnothing$. Note that if $v, w \in I_j - I_{j+1}$ for $j < m$, with w a child of v, then $|A_{vw}| \geqq n/2 + 1$, so $E_v T_w \geqq n$. Similarly, for any child w_m of i_m, $|A_{w_m, i_m}| \leqq (n-1)/2$, since otherwise we could have extended our chain. Thus, for $v, w \in I_m$, w a child of v, $|A_{vw}| \geqq |A_{i_m, w_m}| = n - |A_{w_m, i_m}| \geqq (n+1)/2$, so $E_v T_w \geqq n$.

Therefore, for $w \in I_j - I_{j+1}$ with $d(i_j, w) \geqq \frac{1}{4} \log_{d_{\max}} n$, $E_i T_w \geqq E_{i_j} T_w \geqq t$. Furthermore, if $k \geqq \sqrt{t}$, then for any $w \in I_k$, $d(i, w) \geqq \sqrt{t}$, so by Corollary 6 $E_i T_w \geqq t$. Thus, the only possible vertices w with $E_v T_w < t$ are those in $I_j - I_{j+1}$, $j < \min(\sqrt{t}, m+1)$, such that $d(i_j, w) < \frac{1}{4} \log_{d_{\max}} n$. There can be at most $(\sqrt{t} + 1) d_{\max}^{1/4 \log_{d_{\max}} n} = (\sqrt{t} + 1) n^{1/4}$ such w, from which the lemma follows. $\square$

THEOREM 8. *For trees, for any* $v \in V$, $E_v C \geqq (1/16 - o(1)) n \log_{d_{\max}} n \ln n$.

Proof. The key lemma applies with $\alpha = 1$ and $\beta = \frac{1}{4} - \Theta(\log \log n / \log n)$. $\square$

COROLLARY 9. *For balanced* k-ary *trees, for any* $v \in V$, $E_v C = \Theta(n \log_k n \ln n)$.

Proof. The lower bound follows from the above theorem. The upper bound follows from Theorem 1 by noting that $E_i T_j \leqq 2n$ for i, j neighbors, so $E_i T_j \leqq 2nd(i, j)$, and the diameter of these trees are at most $2 \log_k n$. $\square$

Remark. Aldous [A5] has since shown that, for balanced k-ary trees, $E_v C \sim 2n \log_k n \ln n$.

6. Application to rapidly mixing walks. We now generalize both the $\Omega(\mu^+ \log n)$ lower bound for Cayley graphs and the $\Omega(n \log n)$ lower bound for rapidly mixing walks [BK], that is graphs where $\tau_2 \leqq n^{1-\delta}$ for $\delta > 0$, where $\tau_2 = 1/(1 - \lambda_2)$. We will need the following well-known lemma, which shows that in $O(\tau_2 \log n)$ time the random walk approaches stationarity.

LEMMA 10. *If all eigenvalues are* $\geqq -\lambda_2$, *then for* $t \geqq (k+2)\tau_2 \ln n$,

$$|P_i[X_t=j]-\pi(j)| \leqq n^{-k}.$$

This can be suitably modified if there are eigenvalues $< -\lambda_2$.

Proof. This follows from the spectral representation given in [K] and the facts that $\pi(i) \geqq n^{-2}$ and $|\lambda_k|^t \leqq \lambda_2^t \leqq e^{-t/\tau_2}$. □

LEMMA 11. *Suppose* $\tau_2 \leqq n^{1-\delta}$ *for* $\delta > 0$. *Then for any* $\varepsilon > 0$ *and any* $v \in V$,

$$E_v C \geqq (1-o(1))\delta \min_{\pi(i)<(1+\varepsilon)/n} \{E_\pi T_i\} \ln n.$$

Proof. We show that the key lemma applies with $\alpha = (1 - o(1))$, $\beta = \delta - o(1)$, and $t = (1 - o(1)) \min_{\pi(i)<(1+\varepsilon)/n}\{E_\pi T_i\}$. We set $V' = \{i: \pi(i) < (1+\varepsilon)/n\}$. Note that $|V - V'| \leqq n/(1+\varepsilon)$, so $|V'| \geqq (1 - 1/(1+\varepsilon))n$.

Now fix i. Let $J = \{j: P_i[T_j \leqq 5\tau_2 \ln n] \geqq 1/\ln n\}$. Then $|J| \leqq 5\tau_2 \ln^2 n$, because the sum $\sum_j P_i[T_j \leqq 5\tau_2 \ln n]$ is at most the expected number of vertices visited in time $5\tau_2 \ln n$, namely $5\tau_2 \ln n$.

But for any $j \in V' - J$, $E_i T_j \geqq (1 - 1/\ln n) E_\rho T_j$, where ρ is the distribution after the first $5\tau_2 \ln n$ steps starting at i. By Lemma 10 and using that $\pi(j) \geqq n^{-2}$, $\rho(j)/\pi(j) \geqq 1 - 1/n$. Thus, $E_\rho T_j \geqq (1 - 1/n) E_\pi T_j$, so $E_i T_j \geqq t$. □

To see that this indeed implies an $\Omega(n \log n)$ lower bound, we show the following.

COROLLARY 12. *Suppose* $\tau_2 \leqq n^{1-\delta}$, $\delta > 0$. *Then for any* $\delta' < \delta$ *and any* $v \in V$, $E_v C \geqq (\delta' + o(1)) n \ln n$.

Proof. It suffices to show that for any $\varepsilon > 0$, if $\pi(i) \leqq (1+\varepsilon)/n$, then $E_\pi T_i \geqq (1 - o(1))n/(1+\varepsilon)$. But this follows from the result in [A4] that

$$E_\pi T_i \geqq (1-\pi(i))^2/\pi(i). \quad \square$$

To see where the improvement comes in, we have the following result giving an $\Omega(\mu^+ \log n)$ lower bound for rapidly mixing walks on vertex transitive graphs, generalizing a similar result given for Cayley graphs in [A1].

THEOREM 13. *Define the average hitting time* $\alpha = \sum_{i,j} \pi(i)\pi(j) E_i T_j$. *Suppose* G *is vertex transitive graph and* $\tau_2 \leqq n^{1-\delta}$. *Then for any* $v \in V$,

$$E_v C \geqq (1-o(1))\delta\alpha \ln n.$$

Moreover, $E_v C \leqq (1 + o(1))\alpha \ln n$, *so* $E_v C = \Theta(\alpha \log n)$.

Proof. For vertex transitive graphs, $E_\pi T_i = \alpha$ for all i (see [A2]). Applying Lemma 11 with any $\varepsilon > 0$ then gives the first part. In general vertex transitive graphs, $\alpha \leqq \mu^+ \leqq 2\alpha$ (see [A2]). We can reduce the constant 2 in our situation by observing that if we walk for $5\tau_2 \ln n$ steps, the probability distribution on the vertices is within $(1 + o(1))$ of stationarity. Thus

$$E_i T_j \leqq 5\tau_2 \ln n + (1+o(1)) E_\pi T_j = (1+o(1))\alpha.$$

The second part then follows from Theorem 1. □

7. Lower bound for slowly mixing walks. It is conjectured that $\Omega(n \log n)$ is a lower bound on the expected cover time of any graph. This has been proved in walks starting from the stationary distribution [A3], but it is still open for walks starting at an arbitrary vertex. We complement the above lower bound for rapidly mixing walks with one for slowly mixing walks. This improves the result in [BK], which gives the same lower bound but starting from stationarity. This leaves the general $\Omega(n \log n)$ lower bound open only for graphs with τ_2 between $n^{1-\delta}$ and $n \log n$.

Broder and Karlin prove their lower bound starting from stationarity by showing the following result.

LEMMA 14 ([BK]). *Let α be the average hitting time defined in Theorem* 13. *Then*

$$\alpha = \sum_{k=2}^{n} \tau_k, \quad \text{where } \tau_k = 1/(1-\lambda_k).$$

We improve their result by showing that the expected time to get to a random vertex is independent of the start vertex.

LEMMA 15. *For any i, $\alpha = \sum_j \pi(j) E_i T_j$.*

Our proof makes use of the following lemma.

LEMMA 16 ([KS]). *Except for a slight modification in the bipartite case, the limits*

$$Z_{ij} = \sum_{n=0}^{\infty} (P_i[X_n = j] - \pi(j))$$

exist, and are finite. Moreover,

$$E_\pi T_j = \frac{Z_{jj}}{\pi(j)},$$

$$E_i T_j = \frac{Z_{jj} - Z_{ij}}{\pi(j)}.$$

Proof of Lemma 15. Using Lemma 16,

$$\sum_j \pi(j) E_i T_j = \sum_j (Z_{jj} - Z_{ij}).$$

Rearranging summations, however, yields

$$\sum_j Z_{ij} = \sum_{n=0}^{\infty} \sum_j (P_i[X_n = j] - \pi(j)) = \sum_{n=0}^{\infty} (1-1) = 0.$$

Thus

$$\sum_j \pi(j) E_i T_j = \sum_j Z_{jj} = \sum_j \pi(j) E_\pi T_j = \alpha,$$

as required. □

We can now prove the following theorem.

THEOREM 17. *For any $v \in V$, $E_v C \geqq \sum_{k=2}^{n} \tau_k$.*

Proof. Lemma 15 implies that for any v, we can pick a w with $E_v T_w \geqq \alpha$. Thus $E_v C \geqq E_v T_w \geqq \alpha$. □

COROLLARY 18. *If $\tau_2 = \Omega(n \log n)$, then for any $v \in V$, $E_v C = \Omega(n \log n)$.*

Proof. The proof follows immediately from Theorem 17. □

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