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Covering Times of Random Walks on Bounded Degree Trees and Other Graphs

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The motivating problem for this paper is to find the expected covering time of a random walk on a balanced binary tree with n vertices. Previous upper bounds for general graphs of $O(|V|\,|E|)^{(1)}$ and $O(|V|\,|E|/d_{\min})^{(2)}$ imply an upper bound of $O(n^2)$. We show an upper bound on general graphs of $O(\Delta\,|E|\log|V|)$, which implies an upper bound of $O(n\log^2 n)$. The previous lower bound was $\Omega(|V|\log|V|)$ for trees.(2) In our main result, we show a lower bound of $\Omega(|V|\,(\log_{d_{\max}}|V|)^2)$ for trees, which yields a lower bound of $\Omega(n\log^2 n)$. We also extend our techniques to show an upper bound for general graphs of $O(\max\{E_\pi T_i\}\log|V|)$.

KEY WORDS: Covering times; random walks; graphs; bounded degree trees; balanced binary tree.

1. INTRODUCTION

The motivating problem for this paper is to find the expected covering time of a random walk on a balanced binary tree with n vertices. Previous upper bounds for general graphs of $O(|V|\,|E|)^{(1)}$ and $O(|V|\,|E|/d_{\min})^{(2)}$ imply an upper bound of $O(n^2)$. We show an upper bound on general graphs of $O(\Delta\,|E|\log|V|)$, which implies an upper bound $O(n\log^2 n)$. The previous lower bound was $\Omega(|V|\log|V|)$ for trees.(2) In our main result, we show a lower bound of $\Omega(|V|\,(\log_{d_{\max}}|V|)^2)$ for trees, which yields a lower bound of $\Omega(n\log^2 n)$. We also extend our techniques to show an upper bound for general graphs of $O(\max\{E_\pi T_i\}\log|V|)$.

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2. NOTATION

Our graph is $G=(V, E)$, V being the vertex set and E the edge set. $L \subseteq V$ is the set of leaves for a tree G. Our random walk is $\{X_n\}$. The corresponding edge process is the sequence $\{(X_n, X_{n+1})\}$ of directed edges traversed.

For the following definitions we assume $u, v \in V$: $P_v(\cdot)$ denotes the probability of $(\cdot)$ in a walk starting at v; $E_v(\cdot)$ denotes the expectation of $(\cdot)$ in a walk starting at v; π is the stationary distribution; $E_\pi(\cdot)$ denotes the expectation of $(\cdot)$ in a walk starting from distribution π; C is the time to cover G; C_v is the time to cover G and then return to v; Δ is the diameter of G; d_v is the degree of v; $d_{\max} = \max_v \{d_v\}$; $d_{\min} = \min_v \{d_v\}$; $d(u, v)$ is the distance between u and v; T_v is the time to first reach v; T_v^+ is the time to first reach v, not including time 0. An excursion at v is a walk beginning at v and ending the first time v is reached again. $N_v(G)$ is the number of excursions at v necessary to cover G. We omit the G if it is not ambiguous.

3. PRELIMINARIES

The following well-known lemmas (see Ref. 1) are basic to the theory of random walks on graphs:

Lemma 1. $\pi(v) = d_v/2\,|E|$.

Lemma 2. $E_v T_v^+ = 1/\pi(v) = 2\,|E|/d_v$.

Lemma 3. For $\{u, v\} \in E$, $E_v T_u + E_u T_v \leqslant 2\,|E|$, with equality if and only if the removal of $\{u, v\}$ disconnects G.

Corollary 1. $E_v T_u + E_u T_v \leqslant 2\,|E|\, d(u, v)$, with equality if G is a tree.

The following is very useful:

Proposition 1. Let S be a stopping time such that $X_S = v$. Then, by renewal theory (see Ref. 3)

$$E_v[\#\text{ visits to } w \text{ before time } S] = \pi(w)\, ES$$

Corollary 2. In a tree,

$$E_v[\#\text{ visits to } v \text{ before visit } w] = d_v d(v, w)$$

Proof. Let S be the first visit to v after visiting w in Proposition 1, and observe that Corollary 1 implies $ES = 2\,|E|\, d(v, w)$. The corollary then follows from Lemma 1. □

Corollary 3. $E_v C_v = (EN_v)(E_v T_v^+)$.

Proof. Because C_v is a stopping time, by Proposition 1,

$$EN_v = E_v[\# \text{ visits to } v \text{ before time } C_v] = \pi(v)\, E_v C_v \qquad \square$$

Remark. $E_v C$ can depend heavily on v for a fixed G, as the following illustrates: Let G be a complete graph on n vertices union a vertex v, where v is connected to only one vertex w in the complete graph. Then it is not hard to see that $E_v C = O(n \log n)$, but for $u \neq v$, $E_u C \geqslant E_u T_v = \Omega(n^2)$.

The quantities $E_v C_v$ are better behaved in that they only differ from each other, and from $\max_v \{E_v C\}$, by a factor of 2:

Lemma 4. $E_v C_v \leqslant E_v T_w + E_w C_w + E_w T_v \leqslant 2 E_w C_w$.

Lemma 5. $\frac{1}{2} \max_v \{E_v C_v\} \leqslant \max_v \{E_v C\} \leqslant \max_v \{E_v C_v\}$.

Proof. The first inequality follows because $E_v C_v \leqslant E_v C + \max_u \{E_u T_v\} \leqslant 2 \max_v \{E_v C\}$. The second is obvious. $\square$

Remark. The quantities $E_v C_v$ can differ by a constant factor, as is evident in the path of length n, where n is even. If m is the middle of the path, and v is an end point of the path, it is not hard to see that $EN_m = \frac{3}{2} n$ and $EN_v = n$. Therefore, by Corollary 3, $E_m C_m = \frac{3}{2} n^2$ and $E_v C_v = 2n^2$.

Lemma 6.[4] There exists a spectral representation, i.e., there exists an orthonormal matrix $U = \{u_{ij}\}$ such that

$$P_i[X_n = j] - \pi(j) = \pi^{-1/2}(i)\, \pi^{1/2}(j) \sum_{k=2}^{|V|} \lambda_k^n u_{ik} u_{jk}$$

Lemma 7.[5] Except for a slight modification in the bipartite case, the limits

$$Z_{ij} = \sum_{n=0}^{\infty} \{P_i[X_n = j] - \pi(j)\}$$

exist, and are finite. Moreover, by applying Proposition 1,

$$E_\pi T_i = \frac{Z_{ii}}{\pi(i)}$$

$$E_j T_i = \frac{Z_{ii} - Z_{ji}}{\pi(i)}$$

4. UPPER BOUNDS

We now show that $E_v C = O(\Delta\,|E| \log |V|)$ and, using the matrix $\{Z_{ij}\}$, extend our results to $E_v C = O(\max_i \{E_\pi T_i\} \log |V|)$.

Proposition 2. $E_v C \leqslant [1 + o(1)] e \ln |V| \max_{i,j} \{E_i T_j\}$.

Proof. We can ignore the fact that some times we will use will not be integers; these can be rigorously handled by considering the continuous time case, or simply by taking the greatest integer. In a walk for time $t = e \max_{i,j} \{E_i T_j\}$, no matter what vertex we start at, the probability of not reaching w is at most $1/e$. Hence, if we walk for time $t' = t(\ln |V| + k)$, we have $\ln |V| + k$ walks of length t, so the probability of missing w is at most $(1/e)^{\ln |V| + k} = 1/e^k\,|V|$. Since this holds for every w, the probability of missing at least one vertex is at most e^{-k}, i.e., this walk covers G with probability at least $1 - e^{-k}$. Therefore, the expected number of walks of length t' needed to cover G is at most $1/(1 - e^{-k})$, i.e., $E_v C \leqslant [1/(1 - e^{-k})](\ln |V| + k) e \max\{E_i T_j\}$. Taking $k = (\ln |V|)^{1/2}$ proves the proposition. □

Remark. Proposition 2 allows us to efficiently compute $\max_v \{E_v C\}$ efficiently to within a factor of $O(\log |V|)$, since also $\max_v \{E_v C\} \geqslant \max_{i,j} \{E_i T_j\}$ and $\max_{i,j} \{E_i T_j\}$ is efficiently computable.

Corollary 4. $E_v C \leqslant [1 + o(1)]\, 2e\,\Delta\,|E| \ln |V|$.

Proof. We substitute the fact that $E_i T_j \leqslant 2\,|E|\,\Delta$ (from Corollary 1) into Proposition 2 to obtain the result. □

Remark. Because $\Delta \leqslant 3\,|V|/d_{\min}$, the previous corollary always comes to within $O(\log |V|)$ of the bound $O(|V|\,|E|/d_{\min})$ in Ref. 2.

Lemma 8.

$$\left[\frac{Z_{ij}}{\pi(j)}\right]^2 \leqslant \frac{Z_{ii}}{\pi(i)} \frac{Z_{jj}}{\pi(j)}$$

Proof. Substituting the spectral representation (Lemma 6) into the definition of Z_{ij} (Lemma 7) and dividing by $\pi(j)$, we get

$$\frac{Z_{ij}}{\pi(j)} = \pi^{-1/2}(i)\,\pi^{-1/2}(j) \sum_{k=2}^{|V|} s_k u_{ik} u_{jk}$$

where $s_k = \sum_{n=0}^{\infty} \lambda_k^n = 1/(1-\lambda_k)$. This is also valid for the bipartite case. The lemma follows by noting that Cauchy–Schwartz implies

$$\left(\sum_{k=2}^{|V|} s_k u_{ik} u_{jk}\right)^2 \leqslant \left(\sum_{k=2}^{|V|} s_k u_{ik}^2\right)\left(\sum_{k=2}^{|V|} s_k u_{jk}^2\right) \qquad \square$$

Theorem 1. For all $v \in V$, $E_v C \leqslant [1+o(1)]\, 2e \ln |V| \max_i \{E_\pi T_i\}$.

Proof. From Lemma 8,

$$\frac{|Z_{ij}|}{\pi(j)} \leqslant \max\left\{\frac{Z_{ii}}{\pi(i)}, \frac{Z_{jj}}{\pi(j)}\right\} = \max\{E_\pi T_i, E_\pi T_j\}$$

Thus,

$$\max_{i,j} \{E_i T_j\} = \max_{i,j} \left\{\frac{Z_{jj} - Z_{ij}}{\pi(j)}\right\} \leqslant 2 \max_i \{E_\pi T_i\}$$

The theorem then follows from Proposition 2. $\square$

Proposition 3. In a tree, $E_v C \leqslant [1+o(1)]\, 2e \ln |L| \max_{i \in L} \{E_\pi T_i\}$.

Proof. A random walk on a tree covers all vertices if and only if it covers all the leaves. The proposition follows by restricting w in the proof of Proposition 2 to the set of leaves L, and noting that the largest $E_i T_j$'s occur for i and j leaves. $\square$

5. LOWER BOUND FOR TREES WITH SMALL DEGREE

We now show that for trees, $E_v C = \Omega(|V| (\log_{d_{\max}} |V|)^2)$. The intuitive idea can be illustrated in the case where G is a balanced binary tree, say of height $h \approx \log_2 |V|$. Let r be the root, and s and t its children. Thus s and t are in turn the roots of subtrees G_1 and G_2, respectively, where each G_i has height $h-1$. Now let $n(h) = EN_r(G)$, so $n(h-1) = EN_s(G_1) = EN_t(G_2)$. Thus, we expect to have to go down directed edges (r, s) and (r, t) approximately $n(h-1)/2$ times each in order to cover both G_1 and G_2. If these numbers were deterministically fixed at $n(h-1)/2$ (which they are not), then we would expect to need $n(h-1) + \Omega([n(h-1)]^{1/2})$ visits to the root in order to satisfy both of these quotas. Thus, roughly, we expect $n(h) = n(h-1) + \Omega([n(h-1)]^{1/2})$, which solves to $n(h) = \Omega(h^2)$. Since the expected length of an excursion is $\Theta(|V|)$, if we show that the correlation does not matter too much, then we will have $E_r C = \Omega(n(h)\,|V|) = \Omega(|V| \log^2 |V|)$.

Definition. The neighboring trees in G of a vertex v are the tree components obtained from G by deleting v and all edges incident to v. The root of a neighboring tree is the vertex of the neighboring tree adjacent to v.

Lemma 9. For all $v \in V$, there exists $r \in V$ with two neighboring trees of size at least $|V|/2d_{\max}$. Moreover, one of these trees is of size at most $|V|/2$ and does not contain v.

Proof. Start with $r = v$, and our current tree as the whole tree. Then repeat the following process: if the current r satisfies the conditions of the lemma, then we are done. Otherwise, move to the largest neighboring tree that is a subtree of the current tree, and consider its root as our new r. If the size of the current tree is k, then the size of the next tree is between $(k-1)/(d_{\max}-1)$ and $k-1$. Thus, at some point our current r has a neighboring tree of size between $|V|/2d_{\max}$ and $|V|/2$, and which does not contain v. Then the current r must have another neighboring tree of size at least $(|V|/2-1)/(d_{\max}-1)$. If $|V| \geqslant 2d_{\max}$, then this is at least $|V|/2d_{\max}$. If $|V| < 2d_{\max}$, the lemma is trivial. □

We now come to our main lemma, where we get the recursion we need, although not in its full generality.

Lemma 10. Suppose r has two neighboring trees G_1 and G_2, with s, t being the neighbors of r in G_1, G_2, respectively, such that $EN_s(G_1) \geqslant (d_s - 1)k$ and $EN_t(G_2) \geqslant (d_t - 1)k$, for some $k \geqslant 4$. (Note that $d_s - 1$ is the degree of s in G_1, and similarly for t.) Then $EN_r(G) \geqslant d_r(k + \frac{1}{6}\sqrt{k})$.

Proof. For neighbors v of r, let R_v be the number of times needed to traverse the directed edge (r, v) in a walk starting at r in order to cover the neighboring tree of r containing v. Then the R_v's are independent. Moreover,

$$ER_s = 1 + \frac{1}{d_s - 1} EN_s(G_1) \geqslant k$$

$$ER_t \geqslant k$$

Now let R_{st} be the sum of times over directed edges (r, s) and (r, t) in order that the individual numbers are at least R_s and R_t, respectively. The basic idea of the lemma is that the number of times down (r, s) given the sum of times down (r, s) and (r, t) is binomially distributed. Thus, by insisting that both of our quotas of R_s and R_t are met, ER_{st} must be an additive factor of $\Omega(\sqrt{k})$ larger than $ER_s + ER_t$. To prove this, note first that

$$R_{st} \geqslant R_s + R_t \tag{5.1}$$

Moreover,

$$E[R_{st} \mid R_s, R_t] \geqslant 2 \max(R_s, R_t) \tag{5.2}$$

Now assume without loss of generality that $R_t \geqslant R_s$. Then after $R_s + R_t$ times over the directed edges (r, s) and (r, t), if the number of times over edge (r, t) is $(R_s + R_t)/2 - x$ for $x > 0$, then we expect to go at least $2x$ more times over (r, s) and (r, t) before time R_{st}. Thus, if $\{S_n\}$ is a random walk on the integer line, then the number of additional times over (r, s) and (r, t) is $2E_0[\max\{0, S_n\}] = E[|S_n|] \geqslant \frac{1}{2}\sqrt{n}$, where $n = R_s + R_t$. Therefore,

$$E[R_{st} \mid R_s, R_t] \geqslant R_s + R_t + \tfrac{1}{2}(R_s + R_t)^{1/2} \tag{5.3}$$

We consider two cases:

Case 1. $P(R_s + R_t > k) > 2/3$. Then from (5.1) and (5.3)

$$E[R_{st}] \geqslant \tfrac{1}{3}(E[R_s] + E[R_t]) + \tfrac{2}{3}(E[R_s] + E[R_t] + \tfrac{1}{2}\sqrt{k}) \geqslant 2k + \tfrac{1}{3}\sqrt{k}$$

Case 2. $P(R_s + R_t > k) \leqslant 2/3$. Then without loss of generality we may assume $P(R_s > k/2) \leqslant 5/6$. Then from (5.2)

$$\begin{aligned} E[R_{st}] &\geqslant 2(E[R_t \mid R_s \leqslant k/2] + E[R_s \mid R_s > k/2]) \\ &\geqslant 2(E[R_s + k/2 \mid R_s \leqslant k/2] + E[R_s \mid R_s > k/2]) \\ &\geqslant 2\left(E[R_s] + \frac{1}{6}\frac{k}{2}\right) \geqslant 2k + \frac{1}{6}k \end{aligned}$$

In either case, for $k \geqslant 4$, we conclude that

$$EN_r \geqslant \frac{d_r}{2} E[R_{st}] \geqslant d_r\left(k + \frac{1}{6}\sqrt{k}\right) \qquad \square$$

The following lemma is useful because it says that we expect to visit vertices outside of G_1 a lot before covering G_1.

Lemma 11. For all v not in G_1,

$$E_r[\#\text{ times visit } v \text{ before covering } G_1] = d_v ER_s$$

Proof. Consider the corresponding edge process. Let S be the first time directed edge (s, r) is traversed after covering G_1. Using an analog of Proposition 1, for any directed edge (v, w),

$$E_{(s,r)}[\# \text{ times visit } (v, w) \text{ before } S] = \frac{1}{2\,|E|} E_r S$$

which is independent of (v, w). Taking $(v, w) = (r, s)$, we see that this quantity is ER_s. Thus, for v not in G_1,

$$\begin{aligned} &E_r[\# \text{ times visit } v \text{ before covering } G_1] \\ &\quad = E_r[\# \text{ times visit } v \text{ before } S] \\ &\quad = \sum_{\{v,w\} \in E} E_{(s,r)}[\# \text{ times visit } (v, w) \text{ before } S] = d_v ER_s \qquad \square \end{aligned}$$

We now generalize Lemma 10 to show that our recursion holds even if v is not the vertex with large neighboring trees.

Lemma 12. Suppose we have the same hypotheses as in Lemma 10. Then for any v, $EN_v(G) \geqslant d_v(k + \frac{1}{12}\sqrt{k})$.

Proof. Suppose without loss of generality that v is not in G_1. We distinguish two cases:

Case 1: $d(v, r) \geqslant \frac{1}{12}\sqrt{k}$. By Corollary 2, $E_v[\#$ times visit v before visit $r] = d_v d(v, r)$. When r is first reached, nothing in G_1 has been visited. Therefore, by Lemma 11, the expected number of additional times we visit v before C is $d_v ER_s \geqslant d_v k$. Thus, $EN_v \geqslant d_v(d(v, r) + k)$.

Case 2: $d(v, r) < \frac{1}{12}\sqrt{k}$. Then

$$\begin{aligned} E_v C_v \geqslant E_r C_r - E_v T_r - E_r T_v &= (EN_r)\frac{2\,|E|}{d_r} - 2\,|E|\; d(v, r) \\ &\geqslant 2\,|E|\;[k + \tfrac{1}{6}\sqrt{k} - d(v, r)] \end{aligned}$$

The lemma then follows because $EN_v = (d_v/2\,|E|)\, E_v C_v$. $\square$

We now evaluate our recursion.

Lemma 13. For all $v \in V$, $EN_v \geqslant (1/590)\, d_v(\log_{2d_{\max}} |V| + 1)^2$.

Proof. We proceed by induction on $\log_{2d_{\max}} |V|$. For the base cases, where $\log_{2d_{\max}} |V| \leqslant 290$, pick w with $d(v, w) \geqslant \Delta/2$. Then using $\Delta \geqslant \log_{2d_{\max}} |V|$ and Corollaries 1 and 3,

$$(EN_v)(E_v T_v^+) = E_v C_v \geqslant E_v T_w + E_w T_v \geqslant \Delta\; |E|$$

Thus, using Lemma 2,

$$EN_v \geqslant \frac{1}{2} d_v \Delta \geqslant \frac{1}{2} d_v \log_{2d_{\max}} |V| \geqslant \frac{d_v}{590} (\log_{2d_{\max}} |V| + 1)^2$$

For the inductive step, we apply Lemma 9 to find two subchildren having size at least $|V|/2d_{\max}$. Using the recursive relation given by Lemma 12,

$$\begin{aligned} EN_v/d_v &\geqslant \frac{1}{590} (\log_{2d_{\max}} |V|)^2 + \frac{1}{12} \frac{1}{\sqrt{590}} (\log_{2d_{\max}} |V|) \\ &\geqslant \frac{1}{590} (\log_{2d_{\max}} |V|)^2 + \frac{1}{292} (\log_{2d_{\max}} |V|) \\ &= \frac{1}{590} (\log_{2d_{\max}} |V|)^2 + \frac{1}{295} (\log_{2d_{\max}} |V|) + \frac{3}{292 \cdot 295} (\log_{2d_{\max}} |V|) \\ &\geqslant \frac{1}{590} (\log_{2d_{\max}} |V|)^2 + \frac{1}{295} (\log_{2d_{\max}} |V|) + \frac{1}{590^2} \\ &= \frac{1}{590} (\log_{2d_{\max}} |V| + 1)^2 \end{aligned}$$

□

We are finally ready for the following theorem.

Theorem 2. $E_v C \geqslant (1/600)\,|V|\,(\log_{2d_{\max}} |V|)^2$.

Proof. We would like to say something like $E_v C = E[\#\text{ excursions}]\, E[\text{time/excursion}]$, but unfortunately the two random variables on the right are correlated. We get around this by getting the number of excursions from one subtree and the time per excursion from the others. Our key tool is Lemma 11.

For $|V| \leqslant 60$, $|V|\,(\log_4 |V|)^2 \leqslant 540$, so we assume $|V| > 60$. By Lemma 9, we find a vertex r with a neighboring tree G_1 with root s, where $|V|/2d_{\max} \leqslant |G_1| \leqslant |V|/2$ and G_1 does not contain v. Now we ignore the part of our walk before we first reach r. Therefore, by Lemmas 11 and 13, for all w not in G_1,

$$\begin{aligned} &E_v[\#\text{ times visit } w \text{ before } C] \\ &\quad \geqslant d_w E R_s = \frac{d_w}{d_s - 1} E N_s(G_1) \\ &\quad \geqslant \frac{d_w}{590} \left(\log_{2d_{\max}} \frac{|V|}{2d_{\max}} + 1 \right)^2 \geqslant \frac{|V|}{600} \frac{d_w}{|E|} (\log_{2d_{\max}} |V|)^2 \end{aligned}$$

The result follows because

$$E_v C \geqslant \sum_{w \in G - G_1} E_v[\# \text{ times visit } w \text{ before } C]$$

and

$$\sum_{w \in G - G_1} \frac{d_w}{|E|} \geqslant 1 \qquad \square$$

6. OTHER LOWER BOUNDS

It would be nice if we could obtain a better bound for a bounded degree tree, given its diameter Δ. If G is a balanced binary tree with a path attached to the root, and v is the end of the path, then it is not hard to see that $E_v C = O(\Delta^2 + |V| \log^2 |V|)$. Thus, the best we can hope for is an additional bound of $\Omega(\Delta^2)$. We do this below for general trees:

Proposition 4. For a tree, $E_v C \geqslant \Delta^2/4$.

Proof. By Ref. 6, for an edge $\{i, j\}$, $E_i T_j = 2|A_{ij}| - 1$, where A_{ij} is the subtree containing i obtained by deleting edge $\{i, j\}$. The proposition follows by considering the path from v to a w with $d(v, w) \geqslant \Delta/2$. $\square$

Remark. From the main result of Ref. 7 it is not hard to see that for general graphs, $E_v C = \Omega(\Delta^2/\log |V|)$.

We can improve the previous result for trees if we are concerned with $\max_v \{E_v C\}$:

Proposition 5. In a tree, $\max_v \{E_v C\} \geqslant |E| \Delta$.

Proof. From Corollary 1 and Lemma 5,

$$\max_v \{E_v C\} \geqslant \tfrac{1}{2} \max_v \{E_v C_v\} \geqslant \tfrac{1}{2} \max_{v, w} \{E_v T_w + E_w T_v\} = |E| \Delta \qquad \square$$

Remark. This is not true for general graphs. For example, in the two-dimensional torus, which has diameter $\sqrt{|V|}$, $\max_v \{E_v C\} = O(|V| \log^2 |V|)$.

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