CS 311H: Discrete Math (Fall 2026)

Problem Set #3

Due: Sep 18, 2026

Problem 1: Prove

Prove that if m and n are integers, and mn is even, then m is even or n is even. Clearly state the proof method you choose to use.

Solution. (Contrapositive) Suppose m and n are odd integers. Therefore, there exists intgers k and l such that $m = 2k + 1$, and $n = 2l + 1$.

Now,

$$\begin{aligned} & mn & \\ = \quad & (2k+1)(2l+1) & \text{substititution} \\ = \quad & 4kl + 2k + 2l + 1 & \text{algebra} \\ = \quad & 2(2kl + k + l) + 1 & \text{algebra} \end{aligned}$$

Since integers are closed under multiplication and addition and k and l are integers, $kl + k + l$ is also an integer. Let $t = kl + k + l$.

Therefore, we have $mn = 2t + 1$, where t is some integer. Therefore, mn is odd.

Therefore, if m and n are integers, and mn is even, then m is even or n is even.

Problem 2: Prove

Prove that for an integer n if n^2 is odd, then n is odd.

Solution. (Contrapositive) Suppose n is not an odd integer. Thus n is an even integer and there exists an integer k such that $n = 2k$.
Now,

$$\begin{array}{lll} & n^2 & \\ = & (2k)^2 & \text{substitution} \\ = & 4k^2 & \text{algebra} \\ = & 2(2k^2) & \text{algebra} \end{array}$$

Since multiplication is closed over integers, $2k^2$ is an integer, we have that $n^2 = 2t$, where $t = 2k^2$. Therefore, we have n^2 is even.

Therefore, the contrapostive, for an integer n, if n^2 is odd then n is odd holds.

Problem 3: Prove

For any integer x, if x is divisible by 3, then x^2 is divisible by 3.

Solution. (Direct) Let x be an integer that is divisible by 3. Since $3 \neq 0$, by the definition of divisibility, we know there exists an integer c such that $x = 3c$.

Now,

$$\begin{aligned} & x^2 & \\ = \; & (3c)^2 & \text{substitution} \\ = \; & 3 \times (3c^2) & \text{algebra} \end{aligned}$$

Since, c is an integer and integers are closed under multiplication, $3c^2$ also an integer. Let this integer be t. Now, since we can write x^2 as $3t$, we can say x^2 is also divisible by 3.

Therefore, for any integer x, if x is divisible by 3, then x^2 is divisible by 3.

Problem 4: Prove by contradiction

Prove that $\sqrt{3}$ is irrational.

Hint: You may use the following result without proving it in your proof "For any integer x, if 3 is a factor of x^2, then 3 is a factor of x".

Solution. (Contradiction.) Assume $\sqrt{3}$ is rational. Therefore, there exists integers p and q, where $q \neq 0$, such that $\sqrt{3} = \frac{p}{q}$.

Assume, that p and q have no common factors.

Now,

$$\begin{aligned} & \sqrt{3} = \tfrac{p}{q} \\ \therefore\quad & 3 = \tfrac{p^2}{q^2} \quad \text{squaring both sides} \\ \therefore\quad & 3q^2 = p^2 \quad \text{algebra} \end{aligned}$$

Since, q is an integer, and integers are closed over multiplication, p^2 is a multiple of 3. From the given theorem, therefore p is also a multiple of 3.

Therefore, let $p = 3k$, where k is some integer.

Therefore, we have

$$\begin{aligned} & 3q^2 = (3k)^2 \\ \therefore\quad & 3q^2 = 9k^2 \quad \text{algebra} \\ \therefore\quad & q^2 = 3k^2 \quad \text{Algebra} \end{aligned}$$

Since, k is an integer, and integers are closed over multiplication, q^2 is a multiple of 3. By the given theorem, we can therefore say q is also a multiple of 3.

Therefore, both p and q have a common factor of 3. This contradicts our assumption that p and q have no common factors. Therefore, $\sqrt{3}$ is irrational.

Problem 5: [x points] Proof by Cases

Prove that for any odd integer n, there exists an integer m such that n^2 can be written as $8m + 1$.

Hint: You may use the result that "for any integer x, there exists an integer t, such that $x = 4t, 4t + 1, 4t + 2$, or $4t + 3$" without proving it.

Additional Hint: If an integer x, can be written as $x = 4t, 4t + 1, 4t + 2$, or $4t + 3$ for some integer t, which of these four forms will be odd? (You do not need to prove this before using it in your proof.) You may want to use these as your cases.

Solution. Suppose n is a particular but arbitrarily chosen odd integer. By the hint, we have that $n = 4t$ or $n = 4t + 1$ or $n = 4t + 2$ or $n = 4t + 3$ for some integer t.

In fact, because n is odd and $4t$ and $4t + 2$ are even, n can only be one of $4t + 1$ or $4t + 3$.

Case 1 ($n = 4t+1$ for some integer t): We must find an integer m such that $n^2 = 8m+1$. Since $n = 4t+1$

$$\begin{aligned}
& (4t+1)^2 & \\
= \;& 16t^2 + 8t + 1 & \text{algebra} \\
= \;& 8\left(2t^2 + t\right) + 1 & \text{algebra}
\end{aligned}$$

Let $m = 2t^2 + t$. Then m is an integer since 2 and t are integers and sums and products of integers are integers. Thus, substituting, $n^2 = 8m + 1$ where m is an integer.

Case 2: ($n = 4q+3$ for some integer q): We must find an integer m such that $n^2 = 8m+1$. Since $n = 4q+3$

$$\begin{aligned}
& (4t+3)^2 & \\
= \;& 16t^2 + 24t + 9 & \text{algebra} \\
= \;& 16t^2 + 24t + (8+1) & \text{algebra} \\
= \;& 8\left(2t^2 + 3t + 1\right) + 1 & \text{algebra}
\end{aligned}$$

Let $m = 2q^2+3q+1$. Then m is an integer since $1, 2, 3$, and q are integers and sums and products of integers are integers. Thus, substituting, $n^2 = 8m + 1$, where m is an integer.

Thus, Cases 1 and 2 show that given any odd integer, whether of the form $4t + 1$ or $4t + 3, n^2 = 8m + 1$ for some integer m.

Problem 6:

Suppose a, b are integers. Prove that $(a-3)b^2$ is even if and only if a is odd or b is even.