

CS 311H: Discrete Math (Fall 2026)

Problem Set #5

Due: Sep 25, 2026

Problem 1:

Determine which of the following equalities hold for all sets A , B , and C . If the equality fails, determine whether the statement becomes true if the symbol $=$ is replaced by $\subseteq$ or $\supseteq$. Prove all claims.

(a) $A \cup (A \cap B) = A$.

Solution. For all sets A and B , $A \cup (A \cap B) = A$.

First, we will prove that $x \in A \cup (A \cap B) \subseteq A$.

Let $x \in A \cup (A \cap B)$. Then, by definition, $x \in A \vee x \in A \cap B$. If $x \in A$ then we are done. But if $x \in A \cap B$, then $x \in A \wedge x \in B$. Therefore, $x \in A$. Hence, $A \cup (A \cap B) \subseteq A$.

Now, we must prove that $A \subseteq A \cup (A \cap B)$.

Let $x \in A$. Therefore, we can say $x \in A \vee x \in (A \cap B)$. Hence, $A \subseteq A \cup (A \cap B)$.

Therefore, $A \cup (A \cap B) = A$.

(b) $A - (A - B) = B$.

Solution. $A - (A - B) \neq B$, but $A - (A - B) \subseteq B$.

Proof:

($\subseteq$) Suppose $x \in A - (A - B)$. Then $x \in A \wedge x \notin (A - B)$, or in other words $x \in A \wedge \neg(x \in A - B)$. Expanding the definition again yields

$$x \in A \wedge \neg(x \in A \wedge x \notin B).$$

Applying De Morgan's law gives

$$x \in A \wedge (x \notin A \vee x \in B),$$

which distributes to

$$(x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B).$$

This is just $\text{False} \vee (x \in A \wedge x \in B)$, which reduces to $x \in A \wedge x \in B$. Finally, we conclude that $x \in B$.

Consider $A = \{1\}$ and $B = \{1, 2\}$. Then

$$\begin{aligned} A - (A - B) &= \{1\} - (\{1\} - \{1, 2\}) \\ &= \{1\} - \emptyset \\ &= \{1\}. \end{aligned}$$

Here, $2 \in B$, but $2 \notin A - (A - B)$.

Therefore, $B \not\subseteq A - (A - B)$.

(c) $A - (B \cup C) = (A - B) \cup (A - C)$.

Solution. $A - (B \cup C) \neq (A - B) \cup (A - C)$ but $A - (B \cup C) \subseteq (A - B) \cup (A - C)$.

Proof:

Suppose $x \in A - (B \cup C)$. Then $x \in A \wedge x \notin (B \cup C)$, or in other words $x \in A \wedge \neg(x \in B \cup C)$. Expanding again gives

$$x \in A \wedge \neg(x \in B \vee x \in C),$$

which by De Morgan's law simplifies to

$$x \in A \wedge (x \notin B \wedge x \notin C).$$

This can be rewritten as $(x \in A \wedge x \notin B) \wedge x \notin C$. From this, we can conclude that $x \in A \wedge x \notin B$, so $x \in A - B$, and thus

$$x \in (A - B) \cup (A - C).$$

Consider $A = \{1, 2, 3\}$, $B = \{2\}$, and $C = \{3\}$. Then $A - (B \cup C)$ is

$$\begin{aligned} \{1, 2, 3\} - (\{2\} \cup \{3\}) &= \{1, 2, 3\} - \{2, 3\} \\ &= \{1\}. \end{aligned}$$

But also, $(A - B) \cup (A - C)$ equals

$$\begin{aligned} (\{1, 2, 3\} - \{2\}) \cup (\{1, 2, 3\} - \{3\}) &= \{1, 3\} \cup \{1, 2\} \\ &= \{1, 2, 3\}. \end{aligned}$$

Note that $2 \in (A - B) \cup (A - C)$, but $2 \notin A - (B \cup C)$.

Therefore, $(A - B) \cup (A - C) \not\subseteq A - (B \cup C)$.

Problem 2: Prove

Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution. We have to prove $\mathcal{P}(A) \subseteq \mathcal{P}(B) \leftrightarrow A \subseteq B$. We can, hence, prove, $(\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \subseteq B) \wedge (A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B))$.

First, we will prove $A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Given, $A \subseteq B$. $\mathcal{P}(A)$ is the power set of A . Let set S be an element of $\mathcal{P}(A)$. Therefore, we have,

$$S \in \mathcal{P}(A).$$

By definition of power sets, $S \subseteq A$. Therefore, for every element $x \in S$, by the definition of subsets, we have

$$x \in S \rightarrow x \in A.$$

Since, $A \subseteq B$, for every element $x \in A$, we have

$$x \in A \rightarrow x \in B.$$

Therefore, for every element $x \in S$, we have

$$x \in S \rightarrow x \in B.$$

Therefore,

$$S \subseteq A \subseteq B.$$

Since, $S \subseteq B$, S must be a member of $\mathcal{P}(B)$. Therefore, $S \in \mathcal{P}(B)$.

Now, for every member S of $\mathcal{P}(A)$, if $A \subseteq B$, we can show that $S \in \mathcal{P}(B)$.

Therefore, if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Now, we prove $\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \subseteq B$.

Let $a \in A$. Therefore $\{a\} \subseteq A$, which gives us $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, by the definition of subset, we have $\{a\} \in \mathcal{P}(B)$. Therefore, $\{a\}$ must be a subset of B . Therefore, $a \in B$. Therefore, for every element $a \in A$, we can show that, if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $a \in B$. Thus, $A \subseteq B$.

Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B) \leftrightarrow A \subseteq B$.

Problem 3: Prove or Disprove

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Solution. Let $A = B = \emptyset$. Therefore, $\mathcal{P}(A) = \mathcal{P}(B) = \mathcal{P}(\emptyset) = \{\emptyset\}$.

We have, $A \times B = \emptyset$. Therefore, $\mathcal{P}(A \times B) = \{\emptyset\}$.

Now, $\mathcal{P}(A) \times \mathcal{P}(B) = \{\emptyset\} \times \{\emptyset\}$. Note that $\emptyset \in \{\emptyset\}$.

Therefore, $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset)\}$.

Therefore, $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$.

Problem 4: Prove using Set Identities

Let A, B , and C be sets. Using the set identities (and only the set identities) show that

$$\left(\overline{(A \cup B) \cap (B \cup C)}\right) \cap \overline{(C \cup A)} = (\overline{A} \cap \overline{B}) \cap \overline{C}.$$

Solution.

$$\begin{aligned} & \left(\overline{(A \cup B) \cap (B \cup C)}\right) \cap \overline{(C \cup A)} \\ = & \text{< DeMorgan's Law } \times 3 > \\ & \left(\left(\overline{A \cup B}\right) \cap \left(\overline{B \cup C}\right)\right) \cap \left(\overline{C \cup A}\right) \\ = & \text{< Associativity } \times 2 > \\ & \left(\left(\overline{A} \cap \overline{B}\right) \cap \left(\overline{B} \cap \overline{C}\right)\right) \cap \left(\overline{C} \cap \overline{A}\right) \\ = & \text{< Idempotent } > \\ & \left(\left(\overline{A} \cap \overline{B}\right) \cap \overline{C}\right) \cap \left(\overline{C} \cap \overline{A}\right) \\ = & \text{< Associativity } \times 2 > \\ & \left(\overline{A} \cap \overline{B}\right) \cap \left(\left(\overline{C} \cap \overline{C}\right) \cap \overline{A}\right) \\ = & \text{< Idempotent } > \\ & \left(\overline{A} \cap \overline{B}\right) \cap \left(\overline{C} \cap \overline{A}\right) \\ = & \text{< Commutativity } > \\ & \left(\overline{A} \cap \overline{B}\right) \cap \left(\overline{A} \cap \overline{C}\right) \\ = & \text{< Associativity } > \\ & \left(\left(\overline{A} \cap \overline{B}\right) \cap \overline{A}\right) \cap \overline{C} \\ = & \text{< Commutativity } \times 2; \text{ Associativity } > \\ & \left(\left(\overline{A} \cap \overline{A}\right) \cap \overline{B}\right) \cap \overline{C} \\ = & \text{< Idempotent } > \\ & \left(\overline{A} \cap \overline{B}\right) \cap \overline{C} \end{aligned}$$

Therefore, $\left(\overline{(A \cup B) \cap (B \cup C)}\right) \cap \overline{(C \cup A)} = (\overline{A} \cap \overline{B}) \cap \overline{C}.$

Problem 5: Symmetric difference—Prove using Set-Builder Notation

The symmetric difference of A and B , denoted by $A \oplus B$ is the set containing those elements in A or B , but not in both A and B .

Show using set-builder notation method (and only the set-builder notation method) that

(a) $A \oplus B = (A \cup B) - (A \cap B)$

Solution.

$$\begin{aligned} & A \oplus B \\ = & \text{ < by definition of symmetric difference >} \\ & \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \\ = & \text{ < by definition of union and intersection >} \\ & \{x \mid (x \in A \cup B) \wedge \neg(x \in A \cap B)\} \\ = & \text{ < by definition does not belong >} \\ & \{x \mid (x \in A \cup B) \wedge (x \notin A \cap B)\} \\ = & \text{ < by definition of difference >} \\ & \{x \mid x \in ((A \cup B) - (A \cap B))\} \\ = & \text{ < by definition >} \\ & (A \cup B) - (A \cap B) \end{aligned}$$

(b) $A \oplus B = (A - B) \cup (B - A)$

Solution.

$$\begin{aligned}
 & A \oplus B \\
 = & \quad < \text{by definition of symmetric difference} > \\
 & \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \\
 = & \quad < \text{DeMorgan's} > \\
 & \{x \mid (x \in A \vee x \in B) \wedge (\neg(x \in A) \vee \neg(x \in B))\} \\
 = & \quad < \text{Distributivity} > \\
 & \{x \mid ((x \in A \vee x \in B) \wedge \neg(x \in A)) \vee ((x \in A \vee x \in B) \wedge \neg(x \in B))\} \\
 = & \quad < \text{Distributivity} > \\
 & \{x \mid (x \in A \wedge \neg(x \in A)) \vee (x \in B \wedge \neg(x \in A)) \vee ((x \in A \wedge \neg(x \in B)) \vee (x \in B \wedge \neg(x \in B)))\} \\
 = & \quad < \text{Contradiction} \times 2 > \\
 & \{x \mid F \vee (x \in B \wedge \neg(x \in A)) \vee ((x \in A \wedge \neg(x \in B)) \vee F)\} \\
 = & \quad < \text{Identity laws} > \\
 & \{x \mid (x \in B \wedge \neg(x \in A)) \vee ((x \in A \wedge \neg(x \in B)))\} \\
 = & \quad < \text{Definition of does not belong} > \\
 & \{x \mid (x \in B \wedge x \notin A) \vee ((x \in A \wedge x \notin B))\} \\
 = & \quad < \text{Definition of difference} > \\
 & \{x \mid (x \in B - A) \vee ((x \in A - B))\} \\
 = & \quad < \text{commutativity} > \\
 & \{x \mid ((x \in A - B) \vee (x \in B - A))\} \\
 = & \quad < \text{Union} > \\
 & (A - B) \cup (B - A)
 \end{aligned}$$