

CS 311H: Discrete Math (Fall 2026)

Problem Set #5

Due: Oct 9, 2026

Problem 1: Recursively-defined set

Give a recursive definition of the set of positive integers that are multiples of 5.

Solution. Basis:

$$5 \in S$$

Recursive Step:

If $x \in S$, and $y \in S$ then $x + y \in S$

Problem 2: Recursively-defined functions

Give a recursive definition of the functions $\max$ and $\min$ so that $\max(a_1, a_2, \dots, a_n)$ and $\min(a_1, a_2, \dots, a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, \dots, a_n$.

Solution. Basis:

$n = 1$: $\max(a_1) = a_1$ and $\min(a_1) = a_1$

$n = 2$:

$$\max(a_1, a_2) = \begin{cases} a_1, & \text{if } a_1 \geq a_2 \\ a_2, & \text{if } a_1 < a_2 \end{cases}$$

$$\min(a_1, a_2) = \begin{cases} a_2, & \text{if } a_1 \geq a_2 \\ a_1, & \text{if } a_1 < a_2 \end{cases}$$

Recursive Step:

$$\begin{aligned} \max(a_1, a_2, \dots, a_{n+1}) &= \max(\max(a_1, a_2, \dots, a_n), a_{n+1}) \\ \min(a_1, a_2, \dots, a_{n+1}) &= \min(\min(a_1, a_2, \dots, a_n), a_{n+1}) \end{aligned}$$

Problem 3: Sums of geometric progressions

Prove using Principles of Mathematical Induction

$$\sum_{i=0}^n ar^i = \frac{ar^{n+1} - a}{r - 1},$$

where n is a non-negative integer.

Solution. Proof:

Base case: $n = 0$

For this case, we must show $\sum_{i=0}^0 ar^i = \frac{ar^{0+1} - a}{r-1}$,

$$\begin{aligned} & \sum_{i=0}^0 ar^i \\ & \text{< definition of summation >} \\ = & ar^0 \\ & \text{< algebra >} \\ = & a \\ & \text{< algebra >} \\ = & a \left(\frac{r-1}{r-1} \right) \\ & \text{< algebra >} \\ = & \frac{ar-a}{r-1} \end{aligned}$$

This proves the base case.

Inductive step: *Inductive Hypothesis (IH):* Assume that the result is true for $n = k$, where $k \geq 0$: $\sum_{i=0}^k ar^i = \frac{ar^{k+1} - a}{r-1}$,

We will show that the result holds for $n = k + 1$: $\sum_{i=0}^{k+1} ar^i = \frac{ar^{(k+1)+1} - a}{r-1}$,

$$\begin{aligned} & \sum_{i=0}^{k+1} ar^i \\ & \text{< split off last term >} \\ = & \sum_{i=0}^k ar^i + ar^{k+1} \\ & \text{< IH >} \\ = & \frac{ar^{k+1} - a}{r-1} + ar^{k+1} \\ & \text{< algebra >} \\ = & \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r-1} \\ & \text{< algebra >} \\ = & \frac{ar^{k+1} - a + ar^{(k+1)+1} - ar^{k+1}}{r-1} \\ & \text{< algebra >} \\ = & \frac{ar^{(k+1)+1} - a}{r-1} \end{aligned}$$

This proves the inductive step.

By the Principle of Mathematical Induction, $\sum_{i=0}^n ar^i = \frac{ar^{n+1} - a}{r-1}$, where n is a non-negative integer.

Problem 4: Divisibility

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Note: You may use the result that for any integer a , where $a \neq 0$, if b is divisible by a , and c is divisible by a , then $mb + nc$ is divisible by a whenever m, n are integers.

Solution. Proof:

Base case: $n = 1$

For this case, we must show $1^3 - 1$ is divisible by 3.

$$\begin{aligned} 1^3 - 1 \\ &< \text{arithmetic} > \\ &= 0 \end{aligned}$$

0 is divisible by 3. Therefore, this proves the base case.

Inductive step: *Inductive Hypothesis (IH):* Assume that $k^3 - k$ is divisible by 3.

We will show that the result holds for $n = k + 1$, that is $(k + 1)^3 - (k + 1)$ is divisible by 3.

$$\begin{aligned} &(k + 1)^3 - (k + 1) \\ &= < \text{algebra} > \\ &\quad \left(k^3 + 3k^2 + 3k + 1 \right) - (k + 1) \\ &= < \text{algebra} > \\ &\quad \left(k^3 - k \right) + 3 \left(k^2 + k \right) \end{aligned}$$

By the IH $k^3 - k$ is divisible by 3. Also, $3(k^2 + k)$ is also divisible by 3. Therefore, $(k^3 - k) + 3(k^2 + k)$ is divisible by 3. This proves the inductive step.

By the Principle of Mathematical Induction, $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Problem 5: Inequalities

Prove using Principles of Mathematical Induction

$$n < 2^n,$$

where n is a positive integer.

Solution. Proof:

Base case: $n = 1$

For this case, we must show $1 < 2^1$,

$$\begin{aligned} & 1 \\ & < \text{arithmetic} > \\ & 2 \\ & = \text{arithmetic} > \\ & 2^1 \end{aligned}$$

This proves the base case.

Inductive step: *Inductive Hypothesis (IH):* Assume that the result is true for $n = k$, where $k \geq 1$: $k < 2^k$. We will show that the result holds for $n = k + 1$: $k + 1 < 2^{k+1}$

$$\begin{aligned} & k + 1 \\ & < \text{IH} > \\ & 2^k + 1 \\ & < \text{since } 1 < 2^k (k \geq 1) > \\ & 2^k + 2^k \\ & = \text{algebra} > \\ & 2^{(k+1)} \end{aligned}$$

This proves the inductive step.

By the Principle of Mathematical Induction, $n < 2^n$, where n is a positive integer.

Problem 6: Summation

Prove using Principles of Mathematical Induction

$$\sum_{i=0}^n 2^i = 2^{n+1} - 1,$$

where n is a non-negative integer.

Solution. Proof:

Base case: $n = 0$

For this case, we must show $\sum_{i=0}^0 2^i = 2^{0+1} - 1$.

$$\begin{aligned} & \sum_{i=0}^0 2^i \\ = & \text{ < definition of summation >} \\ & 2^0 \\ = & \text{ < arithmetic >} \\ & 1 \\ = & \text{ < arithmetic >} \\ & 2 - 1 \\ = & \text{ < arithmetic >} \\ & 2^{0+1} - 1 \end{aligned}$$

This proves the base case.

Inductive step: *Inductive Hypothesis (IH):* Assume that the result is true for k , where $k \geq 0$: $\sum_{i=0}^k 2^i = 2^{k+1} - 1$.

We will show that the result holds for $k + 1$: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$.

$$\begin{aligned} & \sum_{i=0}^{k+1} 2^i \\ = & \text{ < split off last term >} \\ & \sum_{i=0}^k 2^i + 2^{k+1} \\ = & \text{ < IH >} \\ & 2^{k+1} - 1 + 2^{k+1} \\ = & \text{ < algebra >} \\ & 2 \times 2^{k+1} - 1 \\ = & \text{ < algebra >} \\ & 2^{(k+1)+1} - 1 \end{aligned}$$

This proves the inductive step.

By the Principle of Mathematical Induction, $\sum_{i=0}^n 2^i = 2^{n+1} - 1$, where n is a non-negative integer.

Problem 7: Divisibility

Prove that when $n \geq 1$, $5^n - 1$ is always divisible by 4.

Solution. Proof:

Base case: $n = 1$ For this case, we must show $(5^n - 1)$ is divisible by 4.

$$\begin{aligned} & (5^1 - 1) \\ &= < \text{arithmetic} > \\ & 4 \end{aligned}$$

4 is divisible by 4.

This proves the base case.

Inductive step: Inductive Hypothesis (IH): Assume that $5^k - 1$ is divisible by 4 for $k \geq 1$. We need to show that $5^{(k+1)} - 1$ is divisible by 4.

$$\begin{aligned} & 5^{(k+1)} - 1 \\ &= < \text{algebra} > \\ & 5 \cdot 5^k - 1 \\ &= < \text{algebra} > \\ & 5 \cdot 5^k - 5 + 4 \\ &= < \text{algebra} > \\ & 5(5^k - 1) + 4 \end{aligned}$$

By I.H., the first term $5(5^k - 1)$ is divisible by 4, and the second term is 4, which is also divisible by 4. Therefore, $5^{(k+1)} - 1$ is divisible by 4.

This proves the inductive step.

By Principles of Mathematical Induction, $5^n - 1$ is always divisible by 4, when $n \geq 1$.

Problem 8: Recursively-defined functions

Consider the function $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(n)$, where

$$\begin{aligned} f(0) &= 10 \\ f(n) &= 2f(n-1) - 3 \end{aligned}$$

and prove that $f(n) \leq 10(2^n)$ for $n \geq 0$.

Solution. Proof: Base Case: $n = 0$

We need to show that $f(0) \leq 10(2^0)$.

$$\begin{aligned} &f(0) \\ = &< \text{recursive-def} > \\ &10 \\ = &< \text{arithmetic} > \\ &10 \cdot 2^0 \end{aligned}$$

Since, we have $f(0) = 10 \cdot 2^0$, we have $f(0) \leq 10 \cdot 2^0$

This proves the base case.

Inductive Case: Inductive Hypothesis (I.H.): Assume that $f(k) \leq 10(2^k)$, for $k \geq 0$.

We need to show that $f(k+1) \leq 10(2^{k+1})$.

$$\begin{aligned} &f(k+1) \\ = &< \text{definition} > \\ &2f(k) - 3 \\ \leq &< \text{arithmetic} > \\ &2f(k) \\ \leq &< \text{I.H} > \\ &2 \cdot 10(2^k) \\ = &< \text{arithmetic} > \\ &10(2^{k+1}) \end{aligned}$$

This proves the inductive step.

By principles of mathematical induction, $f(n) \leq 10(2^n)$ for $n \geq 0$.