

CS 311H: Discrete Math (Fall 2026)

Problem Set #6

Due: Oct 16, 2026

Problem 1: Stamps

Using Strong Induction, prove that any amount of postage more than 7 cents can be paid for with a combination of 3 and 5 cent stamps.

Solution. Proof:

Base case: $n = 8$

We can obtain 8-cent postage with a 3-cent stamp 5-cent stamp.

$n = 9$ We can obtain 9-cent postage with 3 3-cent stamps.

$n = 10$ We can obtain 10-cent postage with 2 5-cent stamps.

This proves the base case.

Inductive step: Inductive Hypothesis (IH): Assume that we can obtain a j -cent postage using 3-cent and 5-cent stamps, where $8 \leq j \leq k$, where $k \geq 10$.

We need to show that we can obtain $k + 1$ cent stamp using 3-cent and 5-cent stamps.

By the I.H. we can create a $(k - 2)$ -cent postage using 3-cent and 5-cent stamps. By adding one 3-cent stamp to that, we obtain a $(k + 1)$ -cent postage.

This proves the inductive case.

By strong induction, any amount of postage more than 7-cents can be paid for with a combination of 3 and 5 cent stamps.

Problem 2: Strong Induction

Show that if $a_1, a_2, \dots, a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted in to their product.

Solution. Proof: We need to prove that for the multiplication of n distinct numbers we need $n - 1$ multiplications *no matter how parentheses are inserted in the product*.

Base Case: $n = 1$

a_1 requires $0 = 1 - 1$ multiplications.

This proves the base case.

Inductive Case: Inductive Hypothesis (I.H.): Assume that for $1 \leq j \leq k$, we need $j - 1$ multiplications to multiply j distinct real numbers.

We need to prove that we need k multiplications to multiply $k + 1$ distinct real numbers.

The last multiplication used to find the $k + 1$ distinct real numbers $a_1, a_2, \dots, a_{k+1}$ is a multiplication of the product of the first j of these numbers for some j , and the product of the last $k + 1 - j$ numbers. (*Note: We have to consider this particular split, since the parenthesis could be inserted anywhere in the product.*)

By the I.H., we need $j - 1$ multiplications to compute the product of the first j terms no matter how the parenthesis were inserted, and $k + 1 - j - 1$ multiplication to compute the product of the last $k + 1 - j$ terms no matter how the parenthesis were inserted.

We need one last multiplication to calculate the product of $k + 1$ numbers.

The total number of multiplications needed to calculate the product of $k + 1$ distinct numbers is

$$= \frac{1 + (j - 1) + (k + 1 - j - 1)}{k} < \text{algebra} >$$

This proves the inductive step.

By strong induction, $P(n)$ is true for all positive integers n .

Problem 3: Strong Induction

Let $a_1 = 2$; $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \geq 3$. Using strong induction, show that $a_n \leq 3^n$ for all positive integers n .

Solution. Proof:

Base case: $n = 1$

For this case, we must show $a_1 \leq 3^1$.

$$\begin{aligned}
 & a_1 \\
 & \quad < \text{definition} > \\
 = & \quad 2 \\
 & \quad < \text{comparison} > \\
 \leq & \quad 3 \\
 & \quad < \text{algebra} > \\
 \leq & \quad 3^1
 \end{aligned}$$

$n = 2$

For this case, we must show $a_2 \leq 3^2$.

$$\begin{aligned}
 & a_2 \\
 & \quad < \text{definition} > \\
 = & \quad 9 \\
 & \quad < \text{comparison} > \\
 \leq & \quad 9 \\
 & \quad < \text{algebra} > \\
 \leq & \quad 3^2
 \end{aligned}$$

This proves the base case.

Inductive step: Inductive Hypothesis (I.H.): Assume that $a_j \leq 3^j$ for all positive integers $j \leq k$, where $k \geq 2$. We have to show that $a_{k+1} \leq 3^{k+1}$.

$$\begin{aligned}
 & a_{k+1} \\
 & \quad < \text{definition} > \\
 = & \quad 2a_k + 3a_{k-1} \\
 & \quad < \text{I.H.} > \\
 \leq & \quad 2 \cdot 3^k + 3 \cdot 3^{k-1} \\
 & \quad < \text{algebra} > \\
 \leq & \quad 2 \cdot 3^k + 3^k \\
 & \quad < \text{algebra} > \\
 \leq & \quad 3 \cdot 3^k \\
 & \quad < \text{algebra} > \\
 \leq & \quad 3^{k+1}
 \end{aligned}$$

This proves the inductive step.

By strong induction, $a_n \leq 3^n$ for all positive integers n .

Problem 4: Strong Induction

The Fibonacci numbers f_n , where $n \geq 0$ are defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, when $n \geq 2$. Using strong induction, prove that for all $n \geq 0$, $f_n = \frac{a^n - b^n}{\sqrt{5}}$, where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$.

Note: a and b are roots of the quadratic equation $x^2 - x - 1 = 0$.

Solution. Proof: Let $P(n)$ be the predicate $f_n = \frac{a^n - b^n}{\sqrt{5}}$.

Base case: $n = 0$. For this case we must show that $f_0 = \frac{a^0 - b^0}{\sqrt{5}}$.

$$\begin{aligned}
 & f_0 \\
 & \quad < \text{definition} > \\
 = & \quad 0 \\
 & \quad < \text{arithmetic} > \\
 = & \quad \frac{0}{\sqrt{5}} \\
 & \quad < \text{algebra} > \\
 = & \quad \frac{a^0 - b^0}{\sqrt{5}}
 \end{aligned}$$

$n = 1$. For this case we must show that $f_1 = \frac{a^1 - b^1}{\sqrt{5}}$.

$$\begin{aligned}
 & f_1 \\
 & \quad < \text{definition} > \\
 = & \quad 1 \\
 & \quad < \text{arithmetic} > \\
 = & \quad \frac{\sqrt{5}}{\sqrt{5}} \\
 & \quad < \text{algebra} > \\
 = & \quad \frac{a^1 - b^1}{\sqrt{5}}
 \end{aligned}$$

This proves the base case.

Inductive step: Inductive Hypothesis (I.H.): Assume that $f_j = \frac{a^j - b^j}{\sqrt{5}}$ for all $0 \leq j \leq k$. We have to show that $f_{k+1} = \frac{a^{k+1} - b^{k+1}}{\sqrt{5}}$.

$$\begin{aligned}
 & f_{k+1} \\
 & \quad < \text{definition} > \\
 = & \quad f_k + f_{k-1} \\
 & \quad < \text{I.H.} > \\
 = & \quad \frac{a^k - b^k}{\sqrt{5}} + \frac{a^{k-1} - b^{k-1}}{\sqrt{5}} \\
 & \quad < \text{algebra} > \\
 = & \quad \frac{a \cdot a^{k-1} - b \cdot b^{k-1}}{\sqrt{5}} + \frac{a^{k-1} - b^{k-1}}{\sqrt{5}} \\
 & \quad < \text{algebra} > \\
 = & \quad \frac{a^{k-1}(a+1) - b^{k-1}(b+1)}{\sqrt{5}} \\
 & \quad < \text{By Note} > \\
 = & \quad \frac{a^{k-1} \cdot a^2 - b^{k-1} \cdot b^2}{\sqrt{5}} \\
 & \quad < \text{algebra} > \\
 = & \quad \frac{a^{k+1} - b^{k+1}}{\sqrt{5}}
 \end{aligned}$$

This proves the inductive step.

By strong induction, for all $n \geq 0$, $f_n = \frac{a^n - b^n}{\sqrt{5}}$, where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$.

Problem 5: Powers of Two

Every positive integer can be written as the sum of distinct powers of two.

Solution. Proof:**Base Case:** $n = 1$

We need to show that 1 can be written as a sum of distinct powers of two.

$$\begin{aligned} &1 \\ &= \langle \text{arithmetic} \rangle \\ &2^0 \end{aligned}$$

Therefore, 1 can be written as a sum of distinct powers of two.

This proves the base case.

Inductive Case: Inductive Hypothesis (I.H.): Assume that for all integers j , where $1 \leq j \leq k$, $k \geq 1$ can be written as a sum of distinct powers of two.

We need to show that $k + 1$, can be written as the sum of distinct powers of two.

We must consider 2 cases, $k + 1$ is even, and $k + 1$ is odd.

Case 1 $k + 1$ is an even number.

Therefore, $\frac{k+1}{2}$ is an integer. Now, $1 \leq \frac{k+1}{2} \leq k$.

Therefore, by the I.H. $\frac{k+1}{2}$ can be written as a sum of distinct powers of two.

Say, $\frac{k+1}{2} = 2^{r_1} + 2^{r_2} + \dots 2^{r_s}$. Therefore, we have,

$$\begin{aligned} \frac{k+1}{2} &= 2^{r_1} + 2^{r_2} + \dots 2^{r_s} \\ \therefore \langle \text{algebra} \rangle \\ k+1 &= 2(2^{r_1} + 2^{r_2} + \dots 2^{r_s}) \\ \therefore \langle \text{algebra} \rangle \\ k+1 &= (2^{r_1+1} + 2^{r_2+1} + \dots 2^{r_s+1}) \end{aligned}$$

Now, since $r_1, r_2, \dots r_s$ are all distinct, $r_1 + 1, r_2 + 1, \dots r_s + 1$ are distinct as well. Therefore, $k + 1$ can be written as a sum of distinct powers of two.

This proves this case.

Case 1 $k + 1$ is odd. Therefore, k is an even number.

Therefore, by the I.H. k can be written as a sum of distinct powers of two. Say $k = 2^{r_1} + 2^{r_2} + \dots 2^{r_s}$.

Since, k is even, none of these powers are zero. Therefore, we can add 2^0 to this sum, which remains a sum of distinct powers of two and is equal to $k + 1$.

This proves this case.

Therefore, we have proven the inductive step.

By strong induction, every positive integer can be written as the sum of distinct powers of two.

Problem 6: Chocolate Bar

Imagine you have a giant Hershey's chocolate bar measuring n inches by m inches (so the bar is a giant grid composed of nm squares of chocolate). We'd like to break the bar into nm individual squares of chocolate. The catch is that at each step, the bar (and subsequent smaller bars) can only be broken vertically or horizontally into two pieces along a grid line. Prove that no matter what strategy you use to break the chocolate into nm pieces, it will always take you $nm - 1$ steps.

Hint: Convert the problem to a single variable and then use Strong Induction.

Solution. *To break up a big chocolate bar, we need one split to make two pieces, and then we can break up the two pieces recursively. This suggests a proof using strong induction on the size of the chocolate bar, where size is measured in chocolate squares. This allows us to convert a two variable problem to a single variable size measured in chocolate squares.*

We must prove the proposition $P(n)$: a chocolate bar of size n can be split into n squares using $n - 1$ splits is true for all $n \geq 1$.

Base Case: $n = 1$

$P(1)$ is true because there is only a single square of chocolate, and $1 - 1 = 0$ splits are required.

This proves the base case.

Inductive Case: Inductive Hypothesis (I.H.): Assume that a chocolate bar of size j , where $1 \leq j \leq k$, $k \geq 1$ can be split into j squares using $j - 1$ splits.

We must now show there is a way to split a chocolate bar of size $k + 1$ with at most k splits.

To do this, first break the chocolate bar of size $k + 1$ into two smaller pieces of size p and q where $p + q = k + 1$. This is certainly possible because the size of the bar is at least two (because k is at least 1).

Now the pieces of sizes p and q are between 1 and k , so by the I.H., breaking these two pieces into single squares requires only $p - 1$ and $q - 1$ splits, respectively.

The total number of splits required to break the bar of size $k + 1$ into single squares is therefore at most $1 + (p - 1) + (q - 1) = p + q - 1 = (k + 1) - 1 = k$.

This proves the inductive step.

By strong induction, a chocolate bar of size n can be split into n squares using $n - 1$ splits is true for all $n \geq 1$

Problem 7: Structural Induction

Show that every well formed formula for compound proposition contains an equal number of left and right parentheses.

Recall: Well-formed formulae for compound propositions is defined recursively as:

Basis: T , F , and s , where s is a propositional variable are well formed formulae.

Recursive Step: If X and Y are well-formed formulae, then $(\neg X)$, $(X \wedge Y)$, $(X \vee Y)$, $(X \rightarrow Y)$, and $(X \leftrightarrow Y)$ are well-formed formulae.

Solution. Proof:

Base case: Each of the formula T , F , and s contain no parentheses, so they contain an equal number of left and right parenthesis.

This proves the base case.

Inductive Step: Inductive Hypothesis (I.H.): Assume that X and Y are well formed formulae, and each contains an equal left and right parentheses.

Let l_X and l_Y be the number of left parenthesis in X and Y respectively, and r_X and r_Y be the number of right parenthesis in X and Y respectively.

Rewriting the I.H. we have $l_X = r_X$ and $l_Y = r_Y$.

We need to show that all the well-formed formulae using the inductive step of the definition, contains an equal number of left and right parenthesis.

The well-formed formulae using the inductive step of the definition are $(\neg X)$, $(X \wedge Y)$, $(X \vee Y)$, $(X \rightarrow Y)$, and $(X \leftrightarrow Y)$.

The number of left parenthesis in the first compound proposition equals $l_X + 1$, and the rest are $l_X + l_Y + 1$. Similarly, The number of right parenthesis in the first compound proposition equals $r_X + 1$, and the rest are $r_X + r_Y + 1$.

By the I.H. $l_X = r_X$ and $l_Y = r_Y$.

Therefore, $l_X + 1 = r_X + 1$ and $l_X + l_Y + 1 = r_X + r_Y + 1$.

This completes the inductive step.

By (structural) induction, we have that all well-formed formulae for compound propositions contains an equal number of left and right parentheses.

Problem 8: Linked Lists

A linked list can be recursively defined as follows

Basis: An empty list denoted by $\langle \rangle$ is a linked list.

Recursive step: If L' is any linked list and x is an arbitrary element then $L = \langle x, L' \rangle$ is also a linked list.

Given a linked list L , I'd like to write a function (method) that counts the number of elements in L .

Consider the following function (method) I have designed:

Algorithm 1 My function

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1: function LEN( $L$ )  $\triangleright$  Hopefully Counts the elements in  $L$ 
2:   if  $L == null$  then
3:     return 0
4:   else
5:      $L' \leftarrow L \rightarrow \text{next}$ 
6:     return  $1 + \text{LEN}(L')$ 
7:   end if
8: end function

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Using Structural Induction, prove that the function (method) $\text{LEN}(L)$ indeed counts the elements in linked list L .

Solution. Proof: $P(L)$: The value returned by $\text{Len}(L)$ equals the actual length (number of elements) of L .

Base case: For $L = null$, we must show that $\text{LEN}(L)$ returns the number of nodes in L .

When $L = null$ there are no nodes. Therefore, L has zero nodes.

Moreover, $\text{LEN}(L)$ returns 0, when $L == null$, which is what we want.

This proves the base case.

Inductive step: Inductive Hypothesis (I.H.): Assume that for linked list L' , $\text{LEN}(L')$ returns the number of nodes in L' .

We need to show that for linked list $L = \langle x, L' \rangle$.

Since $L = \langle x, L' \rangle$, the number of nodes in L is one more than the number of nodes in L' .

Now, since L is not empty. Therefore, the code will take the else branch and return 1 plus the value that the $\text{LEN}(L')$ returns. By the IH, we know that $\text{LEN}(L')$ correctly returns the number of nodes in L' . This is actually the actual length of L . Therefore, $\text{LEN}(L)$ correctly returns the number of nodes in L .

This proves the inductive step.

Therefore, by structural induction, the function (method) $\text{LEN}(L)$ indeed counts the number of nodes in Linked List L .

Problem 9: Full Binary Trees

If T is a full binary tree, then $n(T) \leq 2^{h(T)+1} - 1$.

Solution.

$$P(T) : n(T) = 2^{h(T)+1} - 1$$

Base case: Consider a full binary tree T consisting of just the root r . For this full binary tree, $n(T) = 1$, and $h(T) = 0$.

$$\begin{aligned} & n(T) \\ & \quad < \text{definition} > \\ & = 1 \\ & \quad < \text{comparision} > \\ & \leq 2^{0+1} - 1 \end{aligned}$$

This proves the base case.

Inductive step: Inductive Hypothesis (I.H.) Assume that if T_1 and T_2 are full binary trees, then $n(T_1) \leq 2^{h(T_1)+1} - 1$ and $n(T_2) \leq 2^{h(T_2)+1} - 1$.

By the recursive definition of full binary tress, $T = T_1 \cdot T_2$, where $n(T) = 1 + n(T_1) + n(T_2)$ and $h(T) = 1 + \max(h(T_1), h(T_2))$.

We need to show that $n(T) \leq 2^{h(T)+1} - 1$.

$$\begin{aligned} & n(T) \\ & \quad < \text{by recursive definition of } n(T) > \\ & = 1 + n(T_1) + n(T_2) \\ & \quad < \text{by IH} > \\ & \leq 1 + 2^{h(T_1)+1} - 1 + 2^{h(T_2)+1} - 1 \\ & \quad < \text{algebra}^1 > \\ & \leq 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 \\ & \quad < \text{algebra}^2 > \\ & \leq 2 \cdot 2^{\max(h(T_1)+1, h(T_2)+1)} - 1 \\ & \quad < \text{algebra}^3 > \\ & \leq 2 \cdot 2^{\max(h(T_1), h(T_2))+1} - 1 \\ & \quad < \text{Recursive definition of } h(T) > \\ & = 2 \cdot 2^{h(T)} - 1 \\ & \quad < \text{algebra} > \\ & = 2^{h(T)+1} - 1 \end{aligned}$$

This completes the inductive step.

By induction, If T is a full binary tree T , then $n(T) \leq 2^{h(T)+1} - 1$.

⁰¹The sum of two number is less than or equal to the 2 times the larger number

¹² $\max(2^x, 2^y) = 2^{\max(x, y)}$

²³ $\max(x + 1, y + 1) = \max(x, y) + 1$